1. Let $|\Theta\Psi\rangle$ denote the result of applying the anti-unitary CPT transformation $\hat{\Theta}$ to some state $|\Psi\rangle$, and likewise for $|\Theta\chi\rangle \equiv \hat{\Theta}|\chi\rangle$, etc. The defining relation expressing the anti-unitarity of $\hat{\Theta}$ is $\langle\Theta\Psi|\Theta\chi\rangle = \langle\Psi|\chi\rangle^*$.

   (a) Let $|\Psi\rangle = |0\rangle$ and $|\chi\rangle = \hat{A}(x)|0\rangle$, so $\langle\Psi|\chi\rangle^* = \langle 0|\hat{A}(x)|0\rangle^*$. Since the vacuum $|0\rangle$ is CPT-invariant, $|\Theta\Psi\rangle = \hat{\Theta}|0\rangle = |0\rangle$. If the operator $\hat{A}(x)$ is CPT-even (upper sign) or CPT-odd (lower sign), then $\hat{\Theta}\hat{A}(x)\hat{\Theta}^{-1} = \pm\hat{A}(-x)^\dagger$ and $|\Theta\chi\rangle = \hat{\Theta}\hat{A}(x)|0\rangle = (\hat{\Theta}\hat{A}(x)\hat{\Theta}^{-1})\hat{\Theta}|0\rangle = \pm\hat{A}(-x)^\dagger|0\rangle$. Hence $\langle\Theta\Psi|\Theta\chi\rangle = \langle 0|[\pm\hat{A}(-x)^\dagger|0\rangle] = \pm\langle 0|\hat{A}(-x)^\dagger|0\rangle = \pm\langle 0|\hat{A}(-x)|0\rangle^*$, and therefore the anti-unitarity relation implies that $\langle 0|\hat{A}(x)|0\rangle^* = \pm\langle 0|\hat{A}(-x)|0\rangle^*$. If $\hat{A}$ is CPT-odd, choose $x = 0$ to find $\langle 0|\hat{A}(0)|0\rangle = -\langle 0|\hat{A}(0)|0\rangle$, implying $\langle 0|\hat{A}(0)|0\rangle = 0$. But the vacuum state is (by assumption) translation invariant, so this implies that $\langle 0|\hat{A}(x)|0\rangle = 0$ for all x.

   (b) Now let $|\Psi\rangle = |0\rangle$ and $|\chi\rangle = [\hat{A}(y), \hat{A}(x)^\dagger]|0\rangle$, and (using the CPT invariance of the vacuum) note that $\hat{\Theta}|\chi\rangle = \hat{\Theta}[\hat{A}(y), \hat{A}(x)^\dagger]\hat{\Theta}^{-1}|0\rangle = [\hat{\Theta}\hat{A}(y)\hat{\Theta}^{-1}, \hat{\Theta}\hat{A}(x)^\dagger\hat{\Theta}^{-1}]|0\rangle = [\hat{A}(-y)^\dagger, \hat{A}(-x)]|0\rangle$. The anti-unitarity condition thus implies that $\langle 0|[\hat{A}(y), \hat{A}(x)^\dagger]|0\rangle^* = \langle 0|[\hat{A}(-y)^\dagger, \hat{A}(-x)]|0\rangle$. Complex conjugating both sides gives $\langle 0|[\hat{A}(y), \hat{A}(x)^\dagger]|0\rangle = -\langle 0|[\hat{A}(-y), \hat{A}(-x)^\dagger]|0\rangle$. (The minus sign is due to Hermitian conjugation flipping to order of operators.) Finally, translation invariance of the vacuum state implies that $\langle 0|[\hat{A}(-y), \hat{A}(-x)^\dagger]|0\rangle = \langle 0|[\hat{A}(x), \hat{A}(y)^\dagger]|0\rangle$ (where $x{+}y$ has been added to both coordinates). So $\langle 0|[\hat{A}(y), \hat{A}(x)^\dagger]|0\rangle = -\langle 0|[\hat{A}(x), \hat{A}(y)^\dagger]|0\rangle$, as claimed. Therefore the vacuum expectation of the commutator $\langle 0|[\hat{A}(y), \hat{A}(x)^\dagger]|0\rangle$, which (by translation invariance) only depends on the difference $x{-}y$, is an odd function of $x{-}y$. The spectral density $\chi(q)$ is the spacetime Fourier transform of this expectation value, and Fourier transforming an odd function of $x{-}y$ gives an odd function of q. Hence the spectral density satisfies $\chi(q) = -\chi(-q)$.

2. Free complex scalar field.

   (a) Start with two equal mass real scalar fields, χ_i with $i = 1, 2$, The standard mode expansions for these fields are $\chi_i(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\underset{\sim}{k})}} \big(\hat{a}_i(\underset{\sim}{k})\, e^{ikx} + \hat{a}_i^\dagger(\underset{\sim}{k})\, e^{-ikx}\big)$, and their corresponding conjugate momenta are $\pi_i(x) = \int \frac{d^3k}{(2\pi)^3} \frac{\sqrt{2\omega(\underset{\sim}{k})}}{2i} \big(\hat{a}_i(\underset{\sim}{k})\, e^{ikx} - \hat{a}_i^\dagger(\underset{\sim}{k})\, e^{-ikx}\big)$. These fields satisfy the canonical equal time commutation relations $[\pi_i(\underset{\sim}{x}), \pi_j(\underset{\sim}{y})] = [\chi_i(\underset{\sim}{x}), \chi_j(\underset{\sim}{y})] = 0$ and $i[\pi_i(\underset{\sim}{x}), \chi_j(\underset{\sim}{y})] = \delta_{ij}\, \delta^3(\underset{\sim}{x}{-}\underset{\sim}{y})$. Now build a complex scalar field by defining $\phi(x) = (\chi_1(x) + i\chi_2(x))/\sqrt{2}$, and $\Pi(x) = (\pi_1(x) + i\pi_2(x))/\sqrt{2}$. These complex fields satisfy the equal time commutation relations

$$
\begin{aligned}
i[\Pi^\dagger(\underset{\sim}{x}), \phi(\underset{\sim}{y})] &= \tfrac{i}{2}\big([\pi_1^\dagger(\underset{\sim}{x}), \chi_1(\underset{\sim}{y})] + [\pi_2^\dagger(\underset{\sim}{x}), \chi_2(\underset{\sim}{y})]\big) = \delta^3(\underset{\sim}{x} - \underset{\sim}{y})\,, \\
i[\Pi(\underset{\sim}{x}), \phi(\underset{\sim}{y})] &= \tfrac{i}{2}\big([\pi_1^\dagger(\underset{\sim}{x}), \chi_1(\underset{\sim}{y})] - [\pi_2^\dagger(\underset{\sim}{x}), \chi_2(\underset{\sim}{y})]\big) = 0\,, \\
[\Pi^\dagger(\underset{\sim}{x}), \Pi(\underset{\sim}{y})] &= [\Pi(\underset{\sim}{x}), \Pi(\underset{\sim}{y})] = 0\,, \\
[\phi^\dagger(\underset{\sim}{x}), \phi(\underset{\sim}{y})] &= [\phi(\underset{\sim}{x}), \phi(\underset{\sim}{y})] = 0\,,
\end{aligned}
$$

   (and their Hermitian conjugates). Inverting the mode expansion to express the creation and annihilation operators in terms of the fields gives $\hat{a}_j(\underset{\sim}{k}) = \int d^3x\, \frac{e^{-ikx}}{\sqrt{2\omega(\underset{\sim}{k})}} \big\{i\pi_j(\underset{\sim}{x}) + \omega(\underset{\sim}{k})\, \chi_j(\underset{\sim}{x})\big\}$.

Insert this, and its conjugate, into the Hamiltonian and simplify:

$$
\begin{aligned}
\hat{H} &= \int \frac{d^3k}{(2\pi)^3}\, \omega(\underset{\sim}{k})\big(\hat{a}_1^\dagger(\underset{\sim}{k})\hat{a}_1(\underset{\sim}{k}) + \hat{a}_2^\dagger(\underset{\sim}{k})\hat{a}_2(\underset{\sim}{k})\big) \\
&= \int d^3x\, d^3y\, \frac{d^3k}{(2\pi)^3}\, e^{i\underset{\sim}{k}(\underset{\sim}{x}-\underset{\sim}{y})}\, \tfrac{1}{2}\Big[\, (-i\pi_1(\underset{\sim}{x}) + \omega(\underset{\sim}{k})\,\chi_1(\underset{\sim}{x}))(i\pi_1(\underset{\sim}{y}) + \omega(\underset{\sim}{k})\,\chi_1(\underset{\sim}{y})) \\
&\qquad\qquad + (-i\pi_2(\underset{\sim}{x}) + \omega(\underset{\sim}{k})\,\chi_2(\underset{\sim}{x}))(i\pi_2(\underset{\sim}{y}) + \omega(\underset{\sim}{k})\,\chi_2(\underset{\sim}{y}))\Big] \\
&= \int d^3x\, d^3y\, \frac{d^3k}{(2\pi)^3}\, e^{i\underset{\sim}{k}(\underset{\sim}{x}-\underset{\sim}{y})}\, \tfrac{1}{2}\Big[\, \pi_1(\underset{\sim}{x})\pi_1(\underset{\sim}{y}) + \pi_2(\underset{\sim}{x})\pi_2(\underset{\sim}{y}) \\
&\qquad\qquad + \omega(\underset{\sim}{k})^2\Big(\chi_1(\underset{\sim}{x})\chi_1(\underset{\sim}{y}) + \chi_2(\underset{\sim}{x})\chi_2(\underset{\sim}{y})\Big) \\
&\qquad\qquad - i\omega(\underset{\sim}{k})\,\Big([\pi_1(\underset{\sim}{x}), \chi_1(\underset{\sim}{y})] + [\pi_2(\underset{\sim}{x}), \chi_2(\underset{\sim}{y})]\Big)\Big], \\
&= \int d^3x\, d^3y\, \tfrac{1}{2}\Big[\, \pi_1(\underset{\sim}{x})\pi_1(\underset{\sim}{y}) + \pi_2(\underset{\sim}{x})\pi_2(\underset{\sim}{y})\Big]\, \delta^3(\underset{\sim}{x}-\underset{\sim}{y}) \\
&\qquad + \tfrac{1}{2}\Big(\chi_1(\underset{\sim}{x})\chi_1(\underset{\sim}{y}) + \chi_2(\underset{\sim}{x})\chi_2(\underset{\sim}{y})\Big)\Big[(-\nabla^2 + m^2)\,\delta^3(\underset{\sim}{x}-\underset{\sim}{y})\Big] - \mathcal{V}\int \frac{d^3k}{(2\pi)^3}\,\omega(\underset{\sim}{k})\,. \\
&= \int d^3x\, \tfrac{1}{2}\Big[\, \pi_1(\underset{\sim}{x})\pi_1(\underset{\sim}{x}) + \pi_2(\underset{\sim}{x})\pi_2(\underset{\sim}{x}) + \nabla\chi_1(\underset{\sim}{x})\cdot\nabla\chi_1(\underset{\sim}{x}) + \nabla\chi_2(\underset{\sim}{x})\cdot\nabla\chi_2(\underset{\sim}{x}) \\
&\qquad\qquad + m^2\chi_1(\underset{\sim}{x})\chi_1(\underset{\sim}{x}) + m^2\chi_2(\underset{\sim}{x})\chi_2(\underset{\sim}{x})\Big] - \mathcal{K} \\
&= \int d^3x\, \Big[\Pi^\dagger(\underset{\sim}{x})\Pi(\underset{\sim}{x}) + \nabla\phi^\dagger(\underset{\sim}{x})\cdot\nabla\phi(\underset{\sim}{x}) + m^2\phi^\dagger(\underset{\sim}{x})\phi(\underset{\sim}{x})\Big] - \mathcal{K}\,.
\end{aligned}
$$

In order to write the cross terms as commutators, the change of variables $\underset{\sim}{x} \leftrightarrow \underset{\sim}{y}$, and $\underset{\sim}{k} \to -\underset{\sim}{k}$ was made in the $\chi_i\pi_i$ terms. Then, in all but the last constant term, the Fourier integral over $\underset{\sim}{k}$ was performed, yielding a delta function, or derivatives thereof. Lastly, the derivatives acting on the delta function were integrated by parts (once in $\underset{\sim}{x}$ and once in $\underset{\sim}{y}$) to reach the final form. The (infinite) additive constant $\mathcal{K} \equiv \mathcal{V}\int (d^3k/(2\pi)^3)\,\omega(\underset{\sim}{k})$ precisely cancels the zero point energy of the infinite set of harmonic oscillator degrees of freedom.

(b) Comparing the initial mode expansions to the alternative mode expansion of the complex field,

$$
\begin{aligned}
\phi(x) &= \int \frac{d^3k}{(2\pi)^3}\, \frac{1}{\sqrt{2\omega(\underset{\sim}{k})}}\big(\hat{a}_+(\underset{\sim}{k})\, e^{ikx} + \hat{a}_-^\dagger(\underset{\sim}{k})\, e^{-ikx}\big), \\
\Pi(x) &= \int \frac{d^3k}{(2\pi)^3}\, \frac{\sqrt{2\omega(\underset{\sim}{k})}}{2i}\big(\hat{a}_+(\underset{\sim}{k})\, e^{ikx} - \hat{a}_-^\dagger(\underset{\sim}{k})\, e^{-ikx}\big),
\end{aligned}
$$

shows that $\hat{a}_+(\underset{\sim}{k}) = [\hat{a}_1(\underset{\sim}{k}) + i\hat{a}_2(\underset{\sim}{k})]/\sqrt{2}$ and $\hat{a}_-(\underset{\sim}{k}) = [\hat{a}_1(\underset{\sim}{k}) - i\hat{a}_2(\underset{\sim}{k})]/\sqrt{2}$. Simple algebra shows that the $\hat{a}_\pm$ also satisfy canonical commutation relations, namely $[\hat{a}_+(\underset{\sim}{k}), \hat{a}_+^\dagger(\underset{\sim}{k}')] = [\hat{a}_-(\underset{\sim}{k}), \hat{a}_-^\dagger(\underset{\sim}{k}')] = (2\pi)^3\,\delta^3(\underset{\sim}{k}-\underset{\sim}{k}')$, with all other commutators vanishing. Inserting the inverse relations, $\hat{a}_1(\underset{\sim}{k}) = (\hat{a}_+(\underset{\sim}{k}) + \hat{a}_-(\underset{\sim}{k}))/\sqrt{2}$,$\hat{a}_2(\underset{\sim}{k}) = (\hat{a}_+(\underset{\sim}{k}) - \hat{a}_-(\underset{\sim}{k}))/\sqrt{2}i$ into the original Hamiltonian gives

$$
\begin{aligned}
\hat{H} &\equiv \int \frac{d^3k}{(2\pi)^3}\, \omega(\underset{\sim}{k})\big\{\hat{a}_1^\dagger(\underset{\sim}{k})\hat{a}_1(\underset{\sim}{k}) + \hat{a}_2^\dagger(\underset{\sim}{k})\hat{a}_2(\underset{\sim}{k})\big\} \\
&= \int \frac{d^3k}{(2\pi)^3}\, \omega(\underset{\sim}{k})\, \tfrac{1}{2}\big\{(\hat{a}_+^\dagger(\underset{\sim}{k}) + \hat{a}_-^\dagger(\underset{\sim}{k}))\,(\hat{a}_+(\underset{\sim}{k}) + \hat{a}_-(\underset{\sim}{k})) + (\hat{a}_+^\dagger(\underset{\sim}{k}) - \hat{a}_-^\dagger(\underset{\sim}{k}))\,(\hat{a}_+(\underset{\sim}{k}) - \hat{a}_-(\underset{\sim}{k}))\big\} \\
&= \int \frac{d^3k}{(2\pi)^3}\, \omega(\underset{\sim}{k})\big\{\hat{a}_+^\dagger(\underset{\sim}{k})\hat{a}_+(\underset{\sim}{k}) + \hat{a}_-^\dagger(\underset{\sim}{k})\hat{a}_-(\underset{\sim}{k})\big\}.
\end{aligned}
$$

This looks just like the original form of the Hamiltonian, except for changing $\hat{a}_i(\underset{\sim}{k}) \to \hat{a}_\pm(\underset{\sim}{k})$. Hence, the Hamiltonian, which in its original form appeared to measure the number of excitations created by the $\hat{a}_i^\dagger(\underset{\sim}{k})$ operators (suitably weighted by the single particle energy $\omega(\underset{\sim}{k})$), can *also* be interpreted as measuring the number of excitations created by the $\hat{a}_\pm^\dagger(\underset{\sim}{k})$ operators (weighted by the same single particle energy).

(c) The states: $\hat{a}_1^\dagger(\underset{\sim}{k})|0\rangle$, $\hat{a}_2^\dagger(\underset{\sim}{k})|0\rangle$, $\hat{a}_+^\dagger(\underset{\sim}{k})|0\rangle$, $\hat{a}_-^\dagger(\underset{\sim}{k})|0\rangle$ are all single particle states, but they are not linearly independent. For every momentum $\underset{\sim}{k}$, there is a two-dimensional space of single particle states which are all degenerate eigenstates of the Hamiltonian. The two states, $\hat{a}_1^\dagger(\underset{\sim}{k})|0\rangle$, $\hat{a}_2^\dagger(\underset{\sim}{k})|0\rangle$, provide one choice of orthonormal basis for the single particle eigenspace; the other pair of states $\hat{a}_+^\dagger(\underset{\sim}{k})|0\rangle$, $\hat{a}_-^\dagger(\underset{\sim}{k})|0\rangle$ are simply a *different*, but equally valid, orthonormal basis. Only two (independent) types of particles exist in this theory, but because the particles are exactly degenerate in mass, there is no meaningful definition of which single particle states are "fundamental" and which are quantum superpositions. Note that this is completely analogous to photon polarization (or electron spin) states — for photons one can use linearly polarized basis states, or circularly polarized states (or for electrons, spin up and down with respect to the $\hat{z}$ axis, or up and down with respect to $\hat{x}$, or any other axis). Note that the total number of particles (as well as all other operators) may be written in either the $1,2$ basis, or the $+,-$ basis,

$$\hat{N} = \int \frac{d^3k}{(2\pi)^3} \left\{ \hat{a}_1^\dagger(\underset{\sim}{k})\hat{a}_1(\underset{\sim}{k}) + \hat{a}_2^\dagger(\underset{\sim}{k})\hat{a}_2(\underset{\sim}{k}) \right\} = \int \frac{d^3k}{(2\pi)^3} \left\{ \hat{a}_+^\dagger(\underset{\sim}{k})\hat{a}_+(\underset{\sim}{k}) + \hat{a}_-^\dagger(\underset{\sim}{k})\hat{a}_-(\underset{\sim}{k}) \right\}.$$

3. Interacting complex scalar field.

(a) For a complex scalar field, local Lorentz invariant operators also invariant under $U(1)$ phase rotations ($\phi \to e^{i\alpha}\phi$) with dimensions up to 4 are:

dimension	$U(1)$ invariant operators
0	1
2	$\phi^*\phi$
4	$(\phi^*\phi)^2$, $(\partial\phi)^* \cdot (\partial\phi)$, $\phi^*(\partial^2\phi)$, $(\partial^2\phi)^*\phi$

But the three listed operators with two derivatives are all equivalent upon integration by parts. So the most general (perturbatively) renormalizable $U(1)$ invariant theory of a single scalar field can be put in the form $\mathcal{L} = (\partial\phi)^* \cdot (\partial\phi) + m^2|\phi|^2 + \lambda|\phi|^4 + \text{const.}$

(b) Converting from a Lagrangian to a Hamiltonian formulation, the complex field ϕ will have a (complex) canonical conjugate Π, satisfying $i[\Pi(\underset{\sim}{x})^\dagger, \phi(\underset{\sim}{y})] = i[\Pi(\underset{\sim}{x}), \phi(\underset{\sim}{y})^\dagger] = \delta^3(\underset{\sim}{x}-\underset{\sim}{y})$, with other commutators vanishing. Hence, the generator of a transformation which rotates the phase of ϕ (and Π) is just $Q = i\int d^3x \left[\Pi(\underset{\sim}{x})^\dagger\phi(\underset{\sim}{x}) - \phi(\underset{\sim}{x})^\dagger\Pi(\underset{\sim}{x})\right]$, since this operator is Hermitian and satisfies $[Q, \phi(\underset{\sim}{x})] = \phi(\underset{\sim}{x})$, which is the infinitesimal form of the $U(1)$ symmetry transformation. The generator Q must be Hermitian so that the exponential $U(\alpha) \equiv e^{i\alpha Q}$ is unitary (for real α). and is Hermitian. To prove that this $U(\alpha)$ does implement the desired finite transformation, one can expand in powers of α to find $U(\alpha)\,\phi(\underset{\sim}{x})\,U(\alpha)^\dagger = \sum_{k=0}^\infty \frac{i\alpha)^k}{k!}\,[Q, \cdots [Q, \phi(\underset{\sim}{x})]\cdots] = \sum_{k=0}^\infty \frac{i\alpha)^k}{k!}\,\phi(\underset{\sim}{x}) = e^{i\alpha}\,\phi(\underset{\sim}{x})$, where the first sum involves a nested commutator where $\phi(\underset{\sim}{x})$ is repeatedly commuted with Q a total of k times. (Alternatively, one can note that $\phi(\underset{\sim}{x};\alpha) \equiv U(\alpha)\,\phi(\underset{\sim}{x})\,U(\alpha)^\dagger$ satisfies the first order differential equation $\frac{d}{d\alpha}\,\phi(\underset{\sim}{x};\alpha) = iU(\alpha)\,[Q, \phi(\underset{\sim}{x})]\,U(\alpha)^\dagger = i\phi(\underset{\sim}{x};\alpha)$ with initial condition $\phi(\underset{\sim}{x};0) = \phi(\underset{\sim}{x})$, whose unique solution is $\phi(\underset{\sim}{x};\alpha) = e^{i\alpha}\,\phi(\underset{\sim}{x})$.) Since every term in the Hamiltonian involves the same number of fields $\phi(x)$ as conjugate fields $\phi(x)^\dagger$ (and the symmetry transformation acts the same at every point in space) it is immediate that $U(\alpha)\,H\,U(\alpha)^\dagger = H$, or equivalently that $\frac{d}{dt}Q = -i[Q,\, H] = 0$. So Q is a conserved charge, and the field transformation has the same form at all times, $U(\alpha)\,\phi(x)\,U(\alpha)^\dagger = e^{i\alpha}\,\phi(x)$. Since $\Pi(x) = \dot\phi(x) = \partial_0\phi(x)$, the Lagrangian form of the conserved charge is $Q = i\int d^3x\,\big[(\partial_0\phi(x))^\dagger\phi(x) - \phi(x)^\dagger\partial_0\phi(x)\big]$. Finally, in terms of the $\hat a_\pm(\underset{\sim}{k})$ creation and annihilation opertors (defined in the previous problem), the conserved charge $Q = -\int \frac{d^3k}{(2\pi)^3}\,(\hat a_+^\dagger \hat a_+ - \hat a_-^\dagger \hat a_-)$. So the conserved charge Q simply counts the net number of a_- particles minus a_+ particles. This result for Q may be derived, somewhat tediously, by inserting the mode expansions for $\phi(x)$ and $\Pi(x) = \partial_0\,\phi(x)$, or more easily by just checking that this result satisfies $[Q,\,\phi(x)] = \phi(x)$, as required. (The presence of an irrelevant addition c-number constant may be ruled out by the observation that Q is odd under a charge conjugation transformation taking $\phi(x) \to \phi(x)^\dagger$ and $\Pi(x) \to \Pi(x)^\dagger$.)

(c) With a single complex scalar field, any $U(1)$ invariant theory is automatically charge conjugation invariant, since Hermiticity of the action, or Hamiltonian, requires that coefficients of every term be real. With two or more complex scalar fields this is no longer the case. For example, if χ and ϕ are complex scalars with the same $U(1)$ charge [so the $U(1)$ symmetry is $\chi \to e^{i\alpha}\chi$ and $\phi \to e^{i\alpha}\phi$], then $i(\chi^*\phi - \chi\phi^*)$ is a real, $U(1)$ and Lorentz invariant term which is odd under charge conjugation.

(d) In $D = 3$ spacetime dimensions, a scalar field ϕ has dimension 1/2, so the operator $(\phi^\dagger\phi)^3$ is now renormalizable. The most general (perturbatively) renormalizable $U(1)$ invariant theory of a single scalar field can be put in the form $\mathcal{L} = (\partial\phi)^*\cdot(\partial\phi) + m^2|\phi|^2 + \lambda|\phi|^4 + \eta|\phi|^6 + \text{const.}$

4. Relativistic scalar scattering.

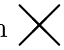
(a) Let $|p,p'\rangle \equiv \hat a^\dagger(\underset{\sim}{p})\,\hat a^\dagger(\underset{\sim}{p}')\,|0\rangle$ be the initial state describing two particles of momenta $\underset{\sim}{p}$ and $\underset{\sim}{p}'$, and $\langle \underset{\sim}{k},\underset{\sim}{k}'|$ the corresponding final state. The first order transition amplitude is

$$
\begin{aligned}
-i\,M &= -i\int_{-\infty}^{\infty} dt\;\langle \underset{\sim}{k},\underset{\sim}{k}'|\hat H_I(t)|\underset{\sim}{p},\underset{\sim}{p}'\rangle \\
&= \frac{-i\lambda}{4!}\int d^4x\;\langle \underset{\sim}{k},\underset{\sim}{k}'|\phi(x)^4|\underset{\sim}{p},\underset{\sim}{p}'\rangle \\
&= \frac{-i\lambda}{4!}\int d^4x\;\frac{d^3q_1}{(2\pi)^3\sqrt{2\omega(\underset{\sim}{q}_1)}}\cdots\frac{d^3q_4}{(2\pi)^3\sqrt{2\omega(\underset{\sim}{q}_4)}} \\
&\qquad\times\langle 0|\hat a(\underset{\sim}{k})\hat a(\underset{\sim}{k}')\left(\prod_{i=1}^{4}\left[\hat a(\underset{\sim}{q}_i)e^{iqx} + \hat a^\dagger(\underset{\sim}{q}_i)e^{-iqx}\right]\right)\hat a^\dagger(\underset{\sim}{p})\hat a^\dagger(\underset{\sim}{p}')|0\rangle \\
&= \frac{-i\lambda}{4!}\,4!\int d^4x\;\frac{e^{i(p+p'-k-k')\cdot x}}{\sqrt{2\omega(\underset{\sim}{p})\,2\omega(\underset{\sim}{p}')\,2\omega(\underset{\sim}{k})\,2\omega(\underset{\sim}{k}')}} \\
&= -i\lambda\,(2\pi)^4\,\delta^4(p+p'-k-k')\,\Big[2\omega(\underset{\sim}{p})\,2\omega(\underset{\sim}{p}')\,2\omega(\underset{\sim}{k})\,2\omega(\underset{\sim}{k}')\Big]^{-1/2}.
\end{aligned}
$$

The expectation value is most easily computed by commuting the $\hat a(\underset{\sim}{k})$ and $\hat a(\underset{\sim}{k}')$ annhilation operators to the right, and the $\hat a^\dagger(\underset{\sim}{p})$ and $\hat a^\dagger(\underset{\sim}{p}')$ creation operators to the left. To obtain a non-zero result (assuming that $\underset{\sim}{p}$ and $\underset{\sim}{p}'$ differ from $\underset{\sim}{k}$ and $\underset{\sim}{k}'$), four different delta function factors (like $(2\pi)^3\delta^3(\underset{\sim}{k} - \underset{\sim}{q}_i)$) must be generated by the commutation through the four field operators. There will be a sum of 4! terms (each involving a product of four delta functions) differing just by a permutation of the $\{q_i\}$. In the final result, as usual, $p^0 \equiv \omega(\underset{\sim}{p})$, etc.

(b) The four-point connected correlation function, to lowest order, is

$$
\Gamma^4_{\text{conn}}(p,p',-k,-k') = -i\lambda\,(2\pi)^4\,\delta^4(p+p'-k-k')\,\tilde G_0(p)\,\tilde G_0(p')\,\tilde G_0(k)\,\tilde G_0(k')\,.
$$

(This is just the momentum space form of the first order diagram  .) To obtain the covariant scattering amplitude, one divides by the product of propagators representing external lines, $\tilde{G}_0(p)\,\tilde{G}_0(p')\,\tilde{G}_0(k)\,\tilde{G}_0(k')$ (thereby "amputating" the external lines), and sets the frequencies equal to their physical values, $p^0 = \omega(\underset{\sim}{p}), \cdots$ (or places the four-momenta "on-shell"). Therefore,

$$-i\mathscr{M} = \Gamma^4_{\text{conn}}\Big|_{\substack{\text{amputated}\\ \text{on-shell}}} = -i\lambda\ (2\pi)^4\,\delta^4(p+p'-k-k')\Big|_{\substack{p^0=\omega(\underset{\sim}{p}),\ p'^0=\omega(\underset{\sim}{p}')\\ k^0=\omega(\underset{\sim}{k}),\ k'^0=\omega(\underset{\sim}{k}')}}$$

$$= -i\,M\Big[2\omega(\underset{\sim}{p})\,2\omega(\underset{\sim}{p}')\,2\omega(\underset{\sim}{k})a\,2\omega(\underset{\sim}{k}')\Big]^{1/2}.$$

(c) The transition rate (straight from Fermi's golden rule) is

$$d\Gamma = \Big|\langle \underset{\sim}{k}, \underset{\sim}{k}'|\hat{H}_I|\underset{\sim}{p}, \underset{\sim}{p}'\rangle\Big|^2\, 2\pi\,\delta(\omega(\underset{\sim}{p})+\omega(\underset{\sim}{p}')-\omega(\underset{\sim}{k})-\omega(\underset{\sim}{k}'))\,\frac{d^3k}{(2\pi)^3}\,\frac{d^3k'}{(2\pi)^3}$$

$$= \frac{\mathcal{V}\,|\tilde{\mathscr{M}}|^2}{2\omega(\underset{\sim}{p})\,2\omega(\underset{\sim}{p}')}\,(2\pi)^4\delta^4(p{+}p'{-}k{-}k')\,\frac{d^3k}{(2\pi)^3 2\omega(\underset{\sim}{k})}\,\frac{d^3k'}{(2\pi)^3 2\omega(\underset{\sim}{k}')}\,.$$

Here, $\tilde{\mathscr{M}}$ is the (covariant) scattering amplitude with the momentum-conservation delta function removed, $\mathscr{M} \equiv \tilde{\mathscr{M}} \times (2\pi)^4\,\delta^4(p{+}p'{-}k{-}k')$. Note that, to lowest order $\tilde{\mathscr{M}}$ is simply the coupling constant λ. As always, the spatial volume $\mathcal{V}$ results from squaring the momentum conserving delta-function. The transition rate equals the cross section times the incident flux, so

$$d\sigma = \frac{d\Gamma}{\text{flux}} = \frac{d\Gamma}{v_{\text{rel}}\,\mathcal{V}} = \frac{\lambda^2}{4EE'v_{\text{rel}}}\,(2\pi)^4\delta^4(p{+}p'{-}k{-}k')\,\frac{d^3k}{(2\pi)^3 2\omega(\underset{\sim}{k})}\,\frac{d^3k'}{(2\pi)^3 2\omega(\underset{\sim}{k}')}\,.$$

Here, $E = \omega(\underset{\sim}{p})$ and $E' = \omega(\underset{\sim}{p}')$ are the energies of the incident particles.
In the center-of-mass frame, $\underset{\sim}{p} + \underset{\sim}{p}' = \underset{\sim}{k} + \underset{\sim}{k}' = 0$, and (since all particles have equal mass) $E = E' = \omega(\underset{\sim}{k}) = \omega(\underset{\sim}{k}')$. The relative velocity of the incident particles is $v_{\text{rel}} = \Big|\frac{\underset{\sim}{p}}{E} - \frac{\underset{\sim}{p}'}{E'}\Big| = \frac{2|\underset{\sim}{p}|}{E}$. Thus,

$$d\sigma_{\text{c.m.}} = \frac{\lambda^2}{4E^2}\,\frac{E}{2|\underset{\sim}{p}|}\,(2\pi)\,\delta(2E{-}2\omega(\underset{\sim}{k}))\,\frac{\underset{\sim}{k}^2\,dk}{4\omega(\underset{\sim}{k})^2}\,\frac{d\Omega}{(2\pi)^3}\,.$$

Or (using the fact that $k\,dk = \omega\,d\omega$)

$$\frac{d\sigma}{d\Omega}\bigg|_{\text{c.m.}} = \frac{\lambda^2}{(4\pi)^2}\,\frac{1}{2}\,\delta(\omega{-}E)\,\frac{\omega\,d\omega}{8E\,\omega^2} = \left(\frac{\lambda}{4\pi}\right)^2\frac{1}{16E^2} = \left(\frac{\lambda}{4\pi}\right)^2\frac{1}{4s}\,,$$

where $s \equiv E^2_{\text{c.m.}} = 4E^2$.

(d) The total cross section equals the integral of the differential cross section over the solid angle representing physically distinct final states. Because the final particles are identical, $|\underset{\sim}{k}, \underset{\sim}{k}'\rangle = |\underset{\sim}{k}', \underset{\sim}{k}\rangle$, this is only *half* the 4π solid angle of a sphere,

$$\sigma_{\text{c.m.}} = \frac{1}{2}\int \frac{d\sigma}{d\Omega}\bigg|_{\text{c.m.}}\,d\Omega = \left(\frac{\lambda}{4\pi}\right)^2\frac{\pi}{2s}\,.$$

(e) Low energy (in the c.m. frame) means that the incoming particles are highly non-relativistic, so that $E = m\,(1 + O(\underset{\sim}{p}^2/m^2))$ or $s = 4m^2 + O(\underset{\sim}{p}^2)$. Therefore,

$$\sigma_{\text{c.m.}} = \left(\frac{\lambda}{4\pi}\right)^2\frac{\pi}{8m^2}\,\big(1 + O\big(\frac{\underset{\sim}{p}^2}{m^2}\big)\big)\,.$$

High energy means that the incoming particles are highly relativistic, so that $E = |\underset{\sim}{p}|\,(1 + O(m^2/\underset{\sim}{p}^2))$ or $s = 4\underset{\sim}{p}^2 + O(m^2)$. Therefore,

$$\sigma_{\text{c.m.}} = \left(\frac{\lambda}{4\pi}\right)^2 \frac{\pi}{8\underset{\sim}{p}^2}\,\Big(1 + O\Big(\frac{m^2}{\underset{\sim}{p}^2}\Big)\Big)\,.$$

The low energy cross section behaves like the square of the interaction strength (λ^2) times the square of the Compton wavelength of the particle ($1/m^2$), while for high energy the cross section scales like $1/\underset{\sim}{p}^2$. Note that the massless limit is the same as the high energy limit. (In other words, the particles are highly relativisitic either if $m \to 0$ for fixed $\underset{\sim}{p}$ or if $\underset{\sim}{p}^2 \to \infty$ for fixed mass.) But in the massless limit, $1/\underset{\sim}{p}^2$ is the only quantity one can form with the dimensions of area, hence the cross section must scale like $1/\underset{\sim}{p}^2$ whenever the mass is negligible compared to the incident momenta.

(f) Assume that p and p' are on-shell collinear momenta. In other words, $p^2{+}m^2 = 0$, $p'^2{+}m'^2 = 0$ and $\underset{\sim}{p}\cdot\underset{\sim}{p}' = -|\underset{\sim}{p}||\underset{\sim}{p}'|$ (minus if the incident particles are oppositely directed). There is no need to assume that the particles have equal mass. Let $E = \omega(\underset{\sim}{p})$ and $E' = \omega(\underset{\sim}{p}')$. Therefore,

$$(p\cdot p')^2 = (-EE' + \underset{\sim}{p}\cdot\underset{\sim}{p}')^2 = (-EE' - |\underset{\sim}{p}||\underset{\sim}{p}'|)^2 = E^2E'^2 + \underset{\sim}{p}^2\underset{\sim}{p}'^2 + 2EE'|\underset{\sim}{p}||\underset{\sim}{p}'|\,.$$

Similarly,

$$\begin{aligned}(EE'v_{\text{rel}})^2 &= E^2E'^2\left(\frac{\underset{\sim}{p}}{E} - \frac{\underset{\sim}{p}'}{E'}\right)^2 = E^2\underset{\sim}{p}'^2 + E'^2\underset{\sim}{p}^2 + 2EE'|\underset{\sim}{p}||\underset{\sim}{p}'| \\ &= E^2\,(E'^2 - m'^2) + (\underset{\sim}{p}'^2 + m'^2)\,\underset{\sim}{p}^2 + 2EE'|\underset{\sim}{p}||\underset{\sim}{p}'| \\ &= E^2E'^2 + \underset{\sim}{p}^2\underset{\sim}{p}'^2 + 2EE'|\underset{\sim}{p}||\underset{\sim}{p}'| - m^2m'^2\,.\end{aligned}$$

Comparing this result with the previous expression reveals that

$$EE'v_{\text{rel}} = \left[(p\cdot p')^2 - m^2m'^2\right]^{1/2}\,,$$

provided p and p' are collinear. Since the right hand side is manifestly Lorentz invariant, this shows that the flux factor $EE'v_{\text{rel}}$ appearing in the denominator of the two-body cross section,

$$d\sigma = \frac{1}{v_{\text{rel}}}\,\frac{\big|\tilde{\mathscr{M}}\big|^2}{2E\,2E'}\,(2\pi)^4\delta^4(p{+}p'{-}k{-}k')\,\frac{d^3k}{(2\pi)^3 2\omega(\underset{\sim}{k})}\,\frac{d^3k'}{(2\pi)^3 2\omega(\underset{\sim}{k}')}$$

is invariant under any Lorentz transformation along the direction of motion of the incident particles. The Feynman rules for the covariant scattering amplitude $\mathscr{M}$ show that it is a completely Lorentz invariant function, as is the four dimensional momentum conserving delta function $\delta^4(P_{\text{in}}{-}P_{\text{out}})$ since the Jacobian of any (proper) Lorentz transformation is unity. Finally, the phase space measure $d^3k/\omega(\underset{\sim}{k})$ is also Lorentz invariant. One way to show this is to write the phase space measure in the form,

$$\frac{d^3p}{(2\pi)^3 2\omega(\underset{\sim}{p})} = \frac{d^4p}{(2\pi)^4}\,(2\pi)\delta(p^2 + m^2)\,\Theta(p^0)\,.$$

(Do the p^0 integral to check.) d^4p is a Lorentz invariant measure and $\delta(p^2 + m^2)$ is manifestly Lorentz invariant. Since a proper Lorentz transformation does not interchange the future and past lightcones, it cannot change the sign of p^0 for any timelike on-shell momentum. Therefore, the step function is also invariant under any proper transformation, and hence so is the entire phase space measure. Since the flux factor is invariant under collinear boosts, and all other ingredients of the cross section are invariant under all proper Lorentz transformations, this implies that the cross section is invariant under Lorentz transformations preserving the directions of the incident particles. Physically the cross section represents an area transverse to the incident direction of motion. A boost along the direction of motion causes Lorentz contraction of longitudinal lengths, but not of transverse lengths, and hence does not change the cross section. The cross section is *not* invariant under boosts which are not parallel to the incident directions, since the area would no longer be orthogonal to the direction of the Lorentz contraction.