1. Wavefunction renormalization.

(a) The two-loop self-energy contribution from the "setting-sun" diagram is

$$F(p) = -(\lambda^2/3!) \int d^d q_1 \, d^d q_2 \, (q_1^2 + m^2)^{-1} (q_2^2 + m^2)^{-1} ((q_1 + q_2 + p)^2 + m^2)^{-1} .$$

The result must be a Lorentz invariant function of the momentum p, so it may be regarded as a function of p^2 and m^2 (with an overall factor of λ^2). As a self-energy contribution, it must also have mass dimension 2. In four dimensions the coupling λ is dimensionless and the two-loop integral is quadratically divergent, so if a (Lorentz invariant) UV momentum cutoff Λ were imposed, the integral could produce UV sensitive terms of three different forms: Λ^2, $m^2 \ln(\Lambda/\mu)$, or $p^2 \ln(\Lambda/\mu)$, with μ some chosen renormalization scale and each form having a coefficient of λ^2 times some pure numnber.

Dimensional analysis alone would permit any of these terms to be multiplied by some non-trivial function of the dimensionless ratio p^2/m^2, but no such function can appear in the UV-sensitive terms which diverge as $\Lambda \to \infty$. Why? Because this UV sensitivity arises from contributions to the integral when the loop momentum is arbitrarily large, much larger than either the mass or external momentum. When evaluating such arbitrarily large momentum contributions, one can expand the propagators in the integrand in a Taylor series in mass and external momentum, multiplied by increasing inverse powers of a loop momentum which progressively decrease (and soon eliminate) any UV-sensitivity.

When using dimensional continuation as the UV regulator, instead of a momentum cutoff Λ, the power-law divergence proportional to Λ^2 is automatically eliminated but each of the logarithmic terms will instead appear as a contribution having a simple pole in dimension proportional to $1/(d-4)$. Hence $F(p) = \lambda^2 (Am^2 + Bp^2)/(d-4) + \text{finite}$.

(b) The momentum-independent contribution proportional to Am^2 may be canceled by suitably adjusting the bare mass-squared, or in other words by writing the bare mass-squared as a renormalized mass-squared plus a series of mass counterterms, $m^2_{\text{bare}} = m^2 + \delta m^2_{(1)} + \delta m^2_{(2)} + \cdots$, where $\delta m^2_{(k)} = c_k \, m^2 \, \lambda^k$ with dimensionless coefficients $\{c_k\}$ containing poles in $d-4$. Treating these mass counterterms as additional perturbations to be expanded in (when generating perturbative diagrammatic contributions), the first such coefficient c_1 may be adjusted to cancel the pole in the one-loop self-energy, while the value $c_2 = -A \, \lambda^2/(d-4)$ for the second coefficient will cancel the momentum-independent pole in in the two-loop self-energy contribution $F(p)$. (Other two-loop self-energy diagrams contain further UV sensitivity which is momentum-independent and which must also be canceled by suitable adjustments of the two-loop mass counterterm, so this is not the final value of c_2.)

Mass-renormalization cannot remove the $B \, p^2$ momentum-dependent pole in $F(p)$, but suitably adjusting the coefficient of the kinetic term can do the job. One must rescale the kinetic term in the action so that it becomes $\frac{1}{2} \, Z(\partial\phi)^2$ with the "wavefunction renormalization" factor Z allowed to deviate from unity and become suitably adjusted order-by-order in perturbation theory, $Z = 1 + \delta Z_{(2)} + \delta Z_{(3)} + \cdots$ where $\delta Z_{(k)} = z_k \, \lambda^k$ with dimensionless coefficients z_k having poles in $d-4$. Treading these wavefunction counterterms as additional perturbations, $\delta Z_{(2)}$ will give rise to a two-loop (i.e., $O(\lambda^2)$) self-energy contribution of simply $\delta Z_{(2)} \, p^2$. So setting $\delta Z_{(2)} = -B \, \lambda^2/(d-4)$ serves to remove the momentum dependent $p^2/(d-4)$ pole in the two-loop self-energy.

(c) Now for the actual work of evaluation:

i. As argued above, the UV-sensitive piece of the two-loop self energy, which cannot be removed by mass renormalization, must be a pure number B times $p^2/(d-4)$. Since this is mass-independent, it is sufficient to work in the massless theory when computing the value of B.

ii. In position space, the two-loop setting-sun diagram involves the Fourier transform of the cube of the free propagator, $F(p) = -\lambda^2 \int d^dx\, e^{-ip\cdot x}\, G_0(x)^3$. Fourier transforms turn produces into convolutions, so this single real-space integral, when expressed in terms of Fourier transformed free propagators, gives the momentum space integral shown initially involving two different d-dimensional loop momentum integrals. In contrast, the position space form of this diagram requires computing a single d-dimensional Fourier transform of a function, $G_0(x)^3$. The free scalar propagator $G_0(x)$ only depends on the Lorentz invariant x^2 and, in the massless limit, is just a power of x^2.

iii. To evaluate the position space form of a massless scalar propagator one must perform the d-dimensional Fourier transform of $1/p^2$:

$$G_0(x) = \int \frac{d^dp}{(2\pi)^d}\,\frac{e^{ip\cdot x}}{p^2} = \int_0^\infty ds \int \frac{d^dp}{(2\pi)^d}\, e^{-sp^2+ip\cdot x} = (2\pi)^{-d}\int_0^\infty ds\, (\pi/s)^{d/2}\, e^{-x^2/(4s)}$$
$$= (4\pi)^{-d/2}\int_0^\infty ds\, s^{-d/2}\, e^{-x^2/(4s)} = \frac{(4/x^2)^{-1+d/2}}{(4\pi)^{d/2}}\,\Gamma(-1+d/2)$$

iv. The integral $I \equiv \int_0^\infty dt\, t^{-4+3d/2}\, e^{-tx^2/4}$ turns into the standard integral reprsentation of a Gamma function after a rescaling of t, showing that $I = (4/x^2)^{-3+3d/2}\,\Gamma(-3+3d/2)$, which immediately gives the claimed identity.

v. Putting things together, the dimensionally-continued Fourier transform of $G_0(x)^3$ is:

$$\begin{aligned}
\widetilde{(G_0^3)}(p) &\equiv (\mu^2)^{(4-d)} \int d^dx\, e^{-ip\cdot x}\, G_0(x)^3 \\
&= \frac{(\mu^2)^{(4-d)}}{(4\pi)^{3d/2}}\,\Gamma(-1+d/2)^3 \int d^dx\, e^{-ip\cdot x}\, (4/x^2)^{-3+3d/2} \\
&= \frac{(\mu^2)^{(4-d)}}{(4\pi)^{3d/2}}\,\frac{\Gamma(-1+d/2)^3}{\Gamma(-3+3d/2)} \int_0^\infty dt\, t^{-4+3d/2} \int d^dx\, e^{-ip\cdot x}\, e^{-t\,(x^2/4)} \\
&= \frac{(\mu^2)^{(4-d)}}{(4\pi)^{d}}\,\frac{\Gamma(-1+d/2)^3}{\Gamma(-3+3d/2)} \int_0^\infty dt\, t^{-4+d}\, e^{-p^2/t} \\
&= \frac{p^2}{(4\pi)^4}\left(\frac{p^2}{4\pi\mu^2}\right)^{d-4} \frac{\Gamma(-1+d/2)^3}{\Gamma(-3+3d/2)}\,\Gamma(3-d)
\end{aligned}$$

vi. To obtain the self-energy contribution, we multiply by $-\lambda^2/6$, since the symmetry factor is $1/3! = 1/6$ and one flips the sign to convert the amputated 1PI diagram into a self-energy contribution. The Laurent expansion of $\widetilde{G_0^3}$ about $d = 4$ begins

$$\widetilde{(G_0^3)}(p) = \frac{p^2}{2(4\pi)^2}\left[\frac{1}{d-4} + \ln\left(\frac{p^2}{4\pi\mu^2}\right) + \gamma_E - \tfrac{13}{4} + O(d-4)\right]$$

vii. Hence, in the massless limit,

$$F(p) = -\tfrac{1}{12}\frac{\lambda^2}{(4\pi)^2}\, p^2 \left[\frac{1}{d-4} + \ln\left(\frac{p^2}{4\pi\mu^2}\right) + +\text{const.}\right],$$

as claimed.

2. Coupled scalar fields.

   (a) Since there are two distinct fields, and two different quartic interaction terms, it is convenient to state Feynman rules using two different types of lines, say solid lines for free propagators of the massive ϕ field and dashed lines for free propagators of the massless χ field, along with two different quartic vertices, one at which four solid lines meet, and one at which two lines are solid and two are dashed. The resulting Feynman rules for this theory are:

      i. Draw relevant diagrams composed of solid lines, dashed lines, and quartic vertices of the two different types.
      ii. Label lines with (directed) momenta.
      iii. Each dashed line labeled with momentum p stands for $1/p^2$. Each solid line labeled with momentum p stands for $1/(p^2+m_0^2)$. (As usual, one may instead use $(p^2+m^2)^{-1}$ with some renormalized mass m as the free propagator, at the cost of introducing quadratic mass counterterm vertices.)
      iv. Each vertex where four solid lines meet stands for a factor of $-\lambda_0$. Each vertex where two solid and two dashed lines meet stands for a factor of $-\eta_0$. (And as usual, one may instead define these vertices as equal to some renormalized couplings, at the cost of introducing additional quartic coupling conterterm vertices.)
      v. Conserve momentum at every vertex and integrate over the resulting undetermined loop momenta.
      vi. Divide by the symmetry factor of the original unlabeled diagram.

   (b) The self-energy for the ϕ field receives one-loop corrections from both a ϕ loop and a χ loop, $\Sigma_\phi(p) = -\frac{\lambda_0}{2}\int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2+m_0^2} - \frac{\eta_0}{2}\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2}$. Both integrals are quadratically divergence, but this cutoff sensitivity can be removed by suitably adjusting the ϕ bare mass. The self-energy for the χ field also has a one-loop correction, $\Sigma_\chi(p) = -\frac{\eta_0}{2}\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2+m_0^2} \propto \Lambda^2$, but no counterterm (or adjustable bare parameter) is avaiable in the theory, as defined, to remove this UV sensitivity. The (amputated) four-point correlator with four ϕ fields receives two one-loop corrections, one with an inner ϕ loop and one with an inner χ loop,

$$\Gamma^{\phi\phi\phi\phi}(p_1\cdots p_4) = -\lambda_0 + \frac{1}{2}\Big[\lambda_0^2\int\frac{d^4q}{(2\pi)^4}\,\frac{1}{q^2+m_0^2}\,\frac{1}{(q-p_1-p_2)^2+m_0^2} + \eta_0^2\int\frac{d^4q}{(2\pi)^4}\,\frac{1}{q^2}\,\frac{1}{(q-p_1-p_2)^2} + \text{permutations}\Big],$$

and the logarithmic sensitivity to the UV cutoff in these integrals can be removed by suitably adjusting the bare coupling λ_0. Similarly, the one-loop (amputated) two-ϕ, two-χ correlator has two contributions,

$$\Gamma^{\phi\phi\chi\chi}(p_1\cdots p_4) = -\eta_0 + \Big[\frac{1}{2}\lambda_0\,\eta_0\int\frac{d^4q}{(2\pi)^4}\,\frac{1}{q^2+m_0^2}\,\frac{1}{(q-p_1-p_2)^2+m_0^2} + \eta_0^2\int\frac{d^4q}{(2\pi)^4}\,\frac{1}{q^2}\,\frac{1}{(q-p_1-p_2)^2+m_0^2} + \text{permutations}\Big],$$

both of which have logarithmic UV sensitivity which can be removed by suitably adjusting the bare coupling η_0. But the four-χ correlator *also* receives a similar one-loop contribution,

$$\Gamma^{\chi\chi\chi\chi}(p_1\cdots p_4) = +\frac{1}{2}\eta_0^2\int\frac{d^4q}{(2\pi)^4}\,\frac{1}{q^2+m_0^2}\,\frac{1}{(q-p_1-p_2)^2+m_0^2} + \text{permutations}\,.$$

This integral is logarithmically sensitive to the UV cutoff Λ, and no counterterm is available to remove this sensitivity. Finally, the vacuum energy is quartically divergent.

(c) Removing the UV sensitivity in χ-self-energy requires introducing a mass term for the χ field, $\frac{1}{2}M_0^2\chi^2$, and suitably adjusting the new bare parameter M_0^2. Similarly, removing the UV sensitivity in $\Gamma^{\chi\chi\chi\chi}$ requires introducing a χ^4 interaction with some new coefficient, say $\frac{1}{4!}\kappa_0\chi^4$, and suitably adjusting κ_0. And finally, the vacuum energy must be adjusted by adding a constant (field independent) vacuum energy counterterm ϵ_0 to the Lagrangian. Putting it all together, if the initial Lagrangian density is generalized to become

$$-\mathcal{L} = \frac{1}{2}\left[(\partial\phi)^2 + m_0^2\,\phi^2\right] + \frac{1}{2}\left[(\partial\chi)^2 + M_0^2\,\chi^2\right] + \tfrac{1}{4!}\,\lambda_0\,\phi^4 + \tfrac{1}{4}\,\eta_0\,\phi^2\chi^2 + \tfrac{1}{4!}\,\kappa_0\,\chi^4 + \epsilon_0\,,$$

then this theory is perturbatively renormalizable (in four dimensiona)

(d) The required adjustments of the bare couplings are

$$\lambda_0(\Lambda) = \lambda(\mu) + \frac{3}{(4\pi)^2}\left(\lambda(\mu)^2 + \eta(\mu)^2\right)\ln(\Lambda/\mu)\,,$$
$$\eta_0(\Lambda) = \eta(\mu) + \frac{1}{(4\pi)^2}\,\eta(\mu)\big(\lambda(\mu) + \kappa(\mu) + 4\eta(\mu)\big)\ln(\Lambda/\mu)\,,$$
$$\kappa_0(\Lambda) = \kappa(\mu) + \frac{3}{(4\pi)^2}\left(\kappa(\mu)^2 + \eta(\mu)^2\right)\ln(\Lambda/\mu)\,,$$

neglecting yet higher order contributions. Differentiating with respect to the arbitrary scale μ gives $0 = \mu\frac{d}{d\mu}\lambda - \frac{3}{(4\pi)^2}(\lambda^2+\eta^2) + \cdots$ where the $\cdots$ represent two-loop or higher contributions which are at least cubic in couplings. Hence

$$\beta_\lambda \equiv \mu\tfrac{d}{d\mu}\lambda = \frac{3}{(4\pi)^2}(\lambda^2+\eta^2) + \cdots\,.$$

Similarly, $\beta_\eta \equiv \mu\frac{d}{d\mu}\,\eta = \frac{\eta}{(4\pi)^2}\left(\lambda+\kappa+4\eta\right) + \cdots$ and $\beta_\kappa \equiv \mu\frac{d}{d\mu}\,\kappa = \frac{3}{(4\pi)^2}\left(\kappa^2+\eta^2\right) + \cdots$. As a check, if $m_0 = M_0$ and $\lambda_0 = \kappa_0 = 3\eta_0$, then this theory has an $O(2)$ symmetry. If the quartic couplings satisfy this $O(2)$ relation at some initial scale, $\lambda(\mu) = \kappa(\mu) = 3\eta(\mu)$, then the above β-functions show that the quartic couplings continue to obey this symmetry relation at all other scales.

3. QED toy model.

We are given the Lagrangian density $-\mathcal{L} = |\partial\phi_e|^2 + m_e^2|\phi_e|^2 + |\partial\phi_\mu|^2 + m_\mu^2|\phi_\mu|^2 + \frac{1}{2}(\partial\chi)^2 + g\,\chi\,(|\phi_e|^2 + |\phi_\mu|^2)$ with m_e positive and much smaller than m_μ.

(a) In the free theory with $g = 0$, the complex scalar field ϕ_μ will destroy a spinless "muon" particle, which will be denoted symbolically as μ^-, or create its antiparticle, an "antimuon" (μ^+). Similarly, ϕ_e will destroy a scalar "electron" (e^-) or create a "positron" (e^+), and the real field χ will create or destroy a scalar "photon" (γ). When the coupling g is non-zero, the $g\,\chi\,|\phi_\mu|^2$ interaction generates non-zero matrix elements between states with a single (bare) photon and states with a (bare) muon and antimuon, as well as non-zero matrix elements between single muon and muon plus photon states. Similarly, the $g\,\chi\,|\phi_e|^2$ interaction generates non-zero matrix elements between single photon and electron-positron pair states, or between electron and electron plus photon states. Note that the interactions do not change the net electron number (*i.e.*, number of electrons minus positrons) or the net muon number.

Despite these non-zero matrix elements, a single electron (or other charged particle) cannot spontaneously emit a photon because 4-momentum cannot be conserved in such a process. (To prove this, note that momentum conservation requires $p = p' + k$ with p and p' on-shell electron 4-momenta, and k an on-shell photon momentum. In order words, $p^2 = p'^2 = -m_e^2$ while and $k^2 = 0$. Squaring both sides of the momentum conservation condition gives $-m_e^2 = -m_e^2 + 2p' \cdot k$ or $p' \cdot k = 0$. But this is impossible for a timelike vector p' and non-vanishing null vector k.) Similarly, a single photon cannot decay to an electron-positron (or muon-antimuon) pair because this would also violate momentum conservation. (The easiest way to prove this is to note that the momentum of a massless photon is null, $k^2 = 0$, while the total momentum of an electron-positron pair satisfies $(p_1 + p_2) \le -(2m_e)^2$, and hence cannot equal a null vector.) More generally, as long as the photon mass is less then twice the electron or muon rest masses, momentum conservation guarantees that the photon must be stable. And momentum conservation combined with the conservation of net electron and muon numbers gurantees that electrons and muons must also be stable. Therefore, the interacting theory will have the same spectrum of stable particles as the $g = 0$ theory: the massless photon and a massive electron and muon, plus their antiparticles.

(b) The (Minkowski space) Feynman rules are:

i. Draw diagrams of the appropriate topology (*i.e.*, number of external lines) composed of directed solid lines (representing the bare electron propagator), directed dashed lines (reprsenting the muon propagator), and undirected wavy lines (representing the photon propagator), and cubic vertices at which a photon line connects to either an incoming and outgoing electron line, or an incoming and outgoing muon line.[1]

ii. Label each line with a four-momentum, and conserve momentum at every vertex.

iii. Each vertex corresponds to a factor of $-ig$.

iv. Each electron line labeled by momentum p corresponds to a factor of $-i/(p^2 + m_e^2 - i\epsilon)$,

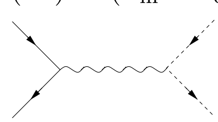
v. Each muon line labeled by momentum p corresponds to a factor of $-i/(p^2 + m_\mu^2 - i\epsilon)$.

vi. Each photon line labeled by momentum k corresponds to a factor of $-i/(k^2 - i\epsilon)$.

vii. Integrate [with measure $d^4 l/(2\pi)^4$] over each undetermined loop momentom l.

viii. Divide by the symmetry factor of the diagram, which is number of permutations exchanging vertices, or lines connecting vertices, which leave the diagram (before labeling momenta) unchanged.

The result is a contribution to the Fourier transformed vacuum expectation value of the time-ordered product of fields corresponding to the given number and types of external lines. To obtain a contribution to the associated covariant scattering amplitude (times $-i$), amputate (*i.e.*, ignore) the external lines, divide by appropriate wavefunction renormalization factors, ignore the overall momentum conserving $(2\pi)^4\, \delta^4(P_{\text{in}} - P_{\text{out}})$, and evaluate on-shell.

(c) There is a single lowest order diagram:  . This gives a covariant scattering amplitude $-i\mathcal{M} = (-ig)^2/[(p_1 + p_2)^2 - i\epsilon]$, where p_1 and p_2 are the momenta of the incoming electron and positron, corresponding to the external propagators attached to (say) the left vertex in the diagram.

(d) The differential cross section $d\sigma = \frac{d^3 p_1'}{(2\pi)^3 (2E_1')} \, \frac{d^3 p_2'}{(2\pi)^3 (2E_2')} \, \frac{|\mathcal{M}|^2}{(2E_1)(2E_2) v_{\text{rel}}} \, (2\pi)^4 \delta^4(p_1 + p_2 - p_1' - p_2')$, where p_1' and p_2' are the momenta of the outgoing muon and antimuon, respectively, and $E_1 = (p_1)^0$, etc. Evaluating this in the center-of-mass frame, we have $\underset{\sim}{p}_1 = -\underset{\sim}{p}_2$, $E_1 = E_2 = \frac{1}{2} E_{\text{c.m.}}$, and $(p_1 + p_2)^2 = -(E_{\text{c.m.}})^2$. Since this cannot vanish, the $i\epsilon$ in the denominator of $\mathcal{M}$ is

[1] Using solid, dashed, and wavy lines to distinguish electron, muon, and photon propagators, respectively, is merely a convention — but one needs to adopt some means of distinguishing the different types of propagators.

irrelevant. The relative velocity $v_{\rm rel} = 2|\underset{\sim}{p}_1|/E_1$ with $|\underset{\sim}{p}_1| = \sqrt{E_1^2 - m_e^2} = \frac{1}{2}\sqrt{E_{\rm c.m.}^2 - 4m_e^2}$. So

$$\begin{aligned}
\sigma_{\rm c.m.} &= \frac{g^4}{(E_{\rm c.m.})^6}\,\frac{E_1}{2|\underset{\sim}{p}_1|}\int \frac{d^3p_1'}{(2\pi)^3(2E_1')}\,\frac{d^3p_2'}{(2\pi)^3(2E_2')}\,(2\pi)^4\delta^4(p_1+p_2-p_1'-p_2') \\
&= \frac{g^4}{(E_{\rm c.m.})^8}\,\frac{E_1}{2|\underset{\sim}{p}_1|}\int\frac{d\Omega}{4\pi^2}\int_0^\infty dp'\,p'^2\;\delta(E_{\rm c.m.}{-}2E') \\
&= \frac{g^4}{(E_{\rm c.m.})^8}\,\frac{E_1}{2|\underset{\sim}{p}_1|}\,\frac{1}{\pi}\int_{m_\mu}^\infty dE'\,E'p'\;\tfrac{1}{2}\delta(E'{-}\tfrac{1}{2}E_{\rm c.m.}) \\
&= \Theta(E_{\rm c.m.}{-}2m_\mu)\,\frac{g^4}{(E_{\rm c.m.})^8}\,\frac{E_1}{2|\underset{\sim}{p}_1|}\,\frac{E'|\underset{\sim}{p}'|}{2\pi}\bigg|_{E'=E_1=\frac{1}{2}E_{\rm c.m.}} \\
&= \Theta(E_{\rm c.m.}{-}2m_\mu)\,\frac{1}{16\pi}\,\frac{g^4}{(E_{\rm c.m.})^6}\left[\frac{E_{\rm c.m.}^2-4m_\mu^2}{E_{\rm c.m.}^2-4m_e^2}\right]^{1/2}.
\end{aligned}$$

Note that $|\underset{\sim}{p}'| \neq |\underset{\sim}{p}_1|$, even though $E' = E_1$, because the initial and final particles have different masses. The step function $\Theta(E_{\rm c.m.}{-}2m_\mu)$ reflects the fact that the delta-function in the last integral cannot be satisfied if $E_{\rm c.m.} < 2m_\mu$.

(e) The threshold energy $E_{\rm min} = 2m_\mu$. For $E_{\rm c.m.} - E_{\rm min} \ll E_{\rm min}$, the cross section $\sigma_{\rm c.m.} \sim \frac{g^4}{16\pi}\,(2m_\mu)^{-7}\sqrt{E_{\rm c.m.}^2{-}(2m_\mu)^2}$, where $m_\mu \gg m_e$ has been used to simplify the form. Consequently, the cross-section rises from threshold proportional to $\sqrt{\Delta E}$ where $\Delta E = E_{\rm c.m.} - 2m_\mu$ is the energy above threshold.

For $E_{\rm c.m.} \gg E_{\rm min}$, the muon (and electron) rest mass may be neglected relative to $E_{\rm c.m.}$ so the cross section simplifies to $\sigma_{\rm c.m.} \sim \frac{g^4}{16\pi}\,(E_{\rm c.m.})^{-6}$. Since the coupling g has dimension 1, the six inverse powers of $E_{\rm c.m.}$ follow just from dimensional analysis, since there is not other relevant energy scale which can provide the needed dimensions.

(f) The fact that real fermions have spin will change the angular dependence of the differential cross section — but this is to be expected. A much bigger difference comes from the fact that the coupling constant g in our toy model has dimensions of energy. This is why the asymptotic high energy cross section falls as $E_{\rm c.m.}^{-6}$. In real QED, the coupling constant e is dimensionless (in natural units). Hence the cross section $\sigma_{e^+e^-\to\mu^+\mu^-} \propto e^4/E_{\rm c.m.}^2 \sim \alpha^2/E_{\rm c.m.}^2$ at high energies. So replacing real (spin one) photons by spinless toy photons changes the high energy behavior of this cross section and will necessarily change the high energy behavior of every other process too. Therefore, the ratio of real QED to toy QED results will not, in general, be close to unity, nor even bounded — the ratio will be arbitrarily large (or small) — so this toy model is a poor approximation to real QED!