Recommended reading: Chapters 37–44 of Srednicki.

1. Dirac Fermions. Consider a theory of free Dirac fermions, with the usual Hamiltonian, $\hat{H} = \int d^3x\, \bar{\psi}(x)\,(\underset{\sim}{\gamma} \cdot \underset{\sim}{\nabla} + m)\,\psi(x)$. Using the standard mode expansion for a Dirac spinor, show that the free time-ordered propagator $S(x)_{\alpha\beta} \equiv \langle 0|\mathscr{T}\left(\psi_\alpha(x)\bar{\psi}_\beta(0)\right)|0\rangle$ is a Green's function for $i(\gamma^\mu\partial_\mu + m)$. What is its Fourier transform $\tilde{S}(p)$? (Make sure your resulting Fourier representation for $S(x)$ is well-defined for Minkowski x.)

2. Wick's theorem for fermions. Consider the time-ordered correlation functions,

$$G_0^{(n,m)}(x_1 \ldots x_n; y_1 \ldots y_m) = \langle 0|\mathscr{T}\left(\psi(x_1)\cdots\psi(x_n)\,\psi^\dagger(y_m)\cdots\psi^\dagger(y_1)\right)|0\rangle$$

in a theory of non-interacting fermions governed by some Hamiltonian $\hat{H} = \int d^3x\; \psi^\dagger(x) h \psi(x)$, where the one-particle hamiltonian h (some spatial differential operator) may have eigenvalues of either sign, and the state $|0\rangle$ is the resulting ground state of the theory. Prove that

$$G_0^{(n,m)}(x_1 \ldots x_n; y_1 \ldots y_m) = \delta_{nm} \sum_\pi \sigma_\pi\, G_0(x_1, y_{\pi_1})\, G_0(x_2, y_{\pi_2}) \cdots G_0(x_n, y_{\pi_n})\,,$$

where the sum runs over all $n!$ permutations π of the set $\{1 \ldots n\}$ and σ_π is the signature of the permutation (+1 for even permutations, −1 for odd). One recommended approach is:

(a) Show that the propagator, $G_0(x,y) \equiv \langle 0|\mathcal{T}(\psi(x)\psi^\dagger(y))|0\rangle$, satisfies the linear equation of motion, $(i\partial_t - h)\, G_0(x,y) = i\delta^4(x-y)$, where $x^0 \equiv t$.

(b) Show that the linear operator $\mathscr{K} \equiv i\partial_t - h$ is invertible *if* all times are analytically continued to a line in the complex time plane which is rotated away (clockwise) from the real axis by an angle $0 < \theta < \pi$ (provided h itself is invertible). Expand $G_0(x,y)$ in an appropriate basis in which one can invert $\mathscr{K}$, and solve for the propagator. Show that the result is the same as that obtained by directly computing the propagator by expanding $\psi(x)$ and $\psi^\dagger(x)$ in terms of creation and annihilation operators for eigenmodes of h. (What is the ground state $|0\rangle$ if h has both positive and negative eigenvalues?)

(c) Derive the corresponding equation of motion for the higher correlation functions $G_0^{(n,m)}$ with arbitrary numbers of operators. Explain why this equation must have a unique well-behaved solution when all times are analytically continued as described above.

(d) Show that the recursive expression $G_0^{(n,m)}(x_1 \ldots x_n; y_1 \ldots y_m) \equiv \sum_{i=1}^m (-)^{n+m-i-1}\, G_0(x_1, y_i) \times G_0^{(n-1,m-1)}(x_2 \ldots x_n; y_1 \ldots y_{i-1}, y_{i+1} \ldots y_m)$ solves the appropriate equation of motion. Iterate this result to prove Wick's theorem for this theory.

(e) Optional. Generalize the above to show that Wick's theorem remains valid at non-zero temperature. What is the finite temperature two-point correlation function $G_0(x,y)$ in this theory?

3. Gamma Matrix Identities. For any four-vector a^μ, let $\not{a}$ be an abbreviation for $\gamma_\mu a^\mu$. Define $\gamma_5 \equiv \gamma^5 = i\,\gamma^0\gamma^1\gamma^2\gamma^3$. Using only the anti-commutation relation satisfied by the gamma matrices:

(a) Prove that γ_5 anticommutes with all the γ_μ, and squares to the identity matrix.

(b) Prove that the trace of any product of an odd number of gamma matrices vanishes.

(c) Evaluate $\mathrm{tr}\,(\not{a}\not{b})$ and $\mathrm{tr}\,(\not{a}\not{b}\not{c}\not{d})$.

(d) How does the number of terms in a trace of k gamma matrices grow with increasing k?

(e) Evaluate (i) $\gamma^\mu\gamma_\mu$, (ii) $\gamma^\mu\not{a}\gamma_\mu$, (iii) $\gamma^\mu\not{a}\not{b}\gamma_\mu$, and (iv) $\gamma^\mu\not{a}\not{b}\not{c}\gamma_\mu$.

4. Majorana representation. Find a gamma matrix representation (in four dimensions) for which $\psi(x)$ is a real field, $\psi(x)^\dagger = \psi(x)$.