1. Bosonic Coherent State Functional Integrals. Standard bosonic coherent states may be defined as $|z\rangle \equiv e^{-\bar{z}z/2} e^{\hat{a}^\dagger z}|0\rangle$ where $z$ is an arbitrary complex number (and $\bar{z} \equiv z^\star$), $\hat{a}^\dagger$ is a canonical bosonic raising operator, and $|0\rangle$ is a normalized harmonic oscillator ground state defined by $\hat{a}|0\rangle = 0$.

   (a) Show that $\hat{a}|z\rangle = z|z\rangle$.
   (b) Compute the overlap $\langle z'|z\rangle$.
   (c) Prove that coherent states satisfy the completeness relation $\hat{1} = \int d\bar{z}\, dz\, |z\rangle\langle z|$, where $d\bar{z}\, dz \equiv d(\mathrm{Re}\, z) d(\mathrm{Im}\, z)/\pi$. [Hint: Write $|z\rangle$ as a sum of normalized harmonic oscillator eigenstates, then do the integral.]
   (d) Prove that $\mathrm{Tr}\, \mathscr{O} = \int d\bar{z}\, dz\, \langle z|\mathscr{O}|z\rangle$.
   (e) Consider an arbitrary Hamiltonian $\hat{H} = h(\hat{a}^\dagger, \hat{a})$ expressed as a (normal ordered) function of $\hat{a}$ and $\hat{a}^\dagger$. By repeatedly inserting the coherent state completeness relation, derive the functional integral representation for the partition function $Z \equiv \mathrm{Tr}\, e^{-\beta H} = \int \mathscr{D}\bar{z}\mathscr{D}z\, e^{-S_E[\bar{z},z]}$ where $S_E[\bar{z}, z] = \int_0^\beta d\tau\, \{\frac{1}{2}\bar{z}(\tau)\overleftrightarrow{\partial}_\tau z(\tau) + h(\bar{z}(\tau), z(\tau))\}$, and the integral is over all complex paths which are periodic with period $\beta$.
   (f) By separating $z(\tau)$ into real and imaginary parts, relate this representation to the "Hamiltonian" path integral derived using alternating position and momentum completeness relations.

2. Fermionic Coherent State Functional Integrals. Generalize the preceding problem to the case of fermionic theories. Let $\hat{b}$ and $\hat{b}^\dagger$ satisfy canonical anticommutation relations. Define fermionic coherent states by the usual formula $|z\rangle \equiv e^{-\bar{z}z/2} e^{\hat{b}^\dagger z}|0\rangle$ where $|0\rangle$ is the state defined by $\hat{b}|0\rangle = 0$, and $z$ and $\bar{z}$ are now regarded as independent generators of a Grassmann algebra. Interpret Hermitian conjugation as interchanging $z$ and $\bar{z}$.

   (a) Show that $\hat{b}|z\rangle = |z\rangle z$.
   (b) Compute the overlap $\langle z'|z\rangle$.
   (c) Using the standard rules of Grassmann integration, prove that fermionic coherent states satisfy the completeness relation $\hat{1} = \int d\bar{z}\, dz\, |z\rangle\langle z|$.
   (d) Prove that $\mathrm{Tr}\, \mathscr{O} = \int d\bar{z}\, dz\, \langle -z|\mathscr{O}|z\rangle$.
   (e) Consider an arbitrary Hamiltonian $\hat{H} = h(\hat{b}^\dagger, \hat{b})$ expressed as a (normal ordered) function of $\hat{b}$ and $\hat{b}^\dagger$. By repeatedly inserting the coherent state completeness relation, derive the functional integral representation for the partition function $Z \equiv \mathrm{Tr}\, e^{-\beta H} = \int \mathscr{D}\bar{z}\mathscr{D}z\, e^{-S_E[\bar{z},z]}$ where $S_E[\bar{z}, z] = \int_0^\beta d\tau\, \{\frac{1}{2}\bar{z}(\tau)\overleftrightarrow{\partial}_\tau z(\tau) + h(\bar{z}(\tau), z(\tau))\}$, and the integral is over all paths for which both $z(\tau)$ and $\bar{z}(\tau)$ are antiperiodic with period $\beta$.

3. Consider a theory of a free Dirac field of mass $m$, defined in a cubic periodic volume of size $L^3$, at (inverse) temperature $\beta$.

   (a) What is the functional integral representation for the partition function? What (precisely) is the appropriate action?
   (b) Show that the free energy is given by the logarithm of a functional determinant, $F = -\beta^{-1} \ln \det_-(\not{\partial} + m)$, where the subscript on $\det_-$ indicates that the operator is acting on the space of antiperiodic functions.
   (c) Express the derivative of the free energy with respect to mass, $\partial F/\partial m^2$, as a sum involving the eigenvalues of $\not{\partial} + m$. (Find these eigenvalues explicitly.)
   (d) Perform the sum over frequencies, take the infinite volume limit, and express $\partial F/\partial m^2$ as a spatial momentum integral with an integrand involving the usual Fermi distribution function.
   (e) Integrate with respect to $m^2$ to find the free energy, and finally differentiate with respect to $\beta$ to find the expectation value of the energy. Do you find the correct result?