1. The transformation $\psi \to e^{i\alpha\gamma_5}\psi$, $\psi^\dagger \to \psi^\dagger e^{-i\alpha\gamma_5}$, with ψ a Dirac spinor field and γ_5 defined to be Hermitian, is referred to as a (continuous) *chiral transformation*.

   (a) What is the explicit form of the chiral transformation when $\alpha = \pi/2$? When $\alpha = \pi$?

   (b) Show that $\bar\psi\partial\!\!\!/\psi$ is invariant under arbitrary chiral transformations.

   (c) Show $\bar\psi\psi$ and $\bar\psi\gamma_5\psi$ transform into linear combinations of each other.

   (d) Explain why the symmetries of a theory of a free massless Dirac fermion include arbitrary chiral transformations. What is the physical meaning of the associated conserved charge?

   (e) Consider a theory with a non-standard mass term, $\mathcal{L} = \bar\psi(\partial\!\!\!/ + m + i\tilde m\gamma_5)\psi$ (with m and $\tilde m$ arbitrary real numbers). Explain why the coefficient of the $\bar\psi\gamma_5\psi$ term must be pure imaginary. Solve this theory by using an appropriate chiral transformation to write it in a more familar form. What is the physical particle mass?

2. Yukawa theory. Consider a theory (in $d = 4$) of a real scalar field ϕ and Dirac fermion ψ, with $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \bar\psi(\partial\!\!\!/ + g\,\phi)\psi + \frac{\lambda}{4!}\phi^4 + (\text{const.})$.

   (a) Explain why this theory is the most general renormalizable theory (up to field rescalings) involving the given fields whose symmetries include discrete γ_5 chiral transformations as well as Lorentz transformations and overall fermion phase rotations. Give examples of additional renormalizable terms which could be added if symmetry is only required under two of these three classes of transformations. In other words, what additional terms could be present if (i) one only demands Lorentz and phase rotation symmetry, or (ii) one only demands Lorentz and chiral symmetry, or (iii) one requires chiral and phase rotation symmetry (but not Lorentz).

   (b) Let Λ denote a UV cutoff. Assume that the scalar mass squared, m^2, is positive. Work out the required one-loop renormalization of m^2, g, and λ.

   (c) Now suppose that m^2 is negative (but λ is positive, as required for stability of the vacuum). What values of ϕ minimize the (scalar contribution to the) Euclidean action? Expand the action about your minima, and clearly state the resulting Feynman rules. Using your modified Feynman rules, work out the one-loop renormalization of the fermion self-energy and the fermion-scalar vertex (*i.e.*, the 1PI part of the $\langle\bar\psi(x)_\alpha\phi(y)\,\psi(z)_\beta\rangle$ three-point function). You should find that the fermion self-energy now requires both mass and wavefunction renormalization. Nevertheless, you should be able to show that suitable adjustment of the parameters in the original Lagrangian still suffice to remove all UV sensitivity from both the fermion self-energy and the vertex function.

3. Gauge indices (*"to index or not to index"*). Let $\mathfrak{g}$ be the Lie algebra corresponding to some compact gauge group $\mathscr{G}$. Let $\{t_a/i\}$, for $a = 1, \cdots, \dim(\mathfrak{g})$ be a set of basis elements for the Lie algebra. These satisfy $[t_a, t_b] = i f_{ab}{}^c t_c$, with $f_{ab}{}^c$ the structure constants of the Lie algebra. In any irreducible representation R, these basis elements may be represented as $\dim(\mathrm{R})\times\dim(\mathrm{R})$ matrices $\{t_a^{\mathrm{R}}/i\}$ (with t_a^{R} Hermitian). One may choose an orthogonal basis, so that $\mathrm{tr}\,(t_a^{\mathrm{R}}\,t_b^{\mathrm{R}}) = \delta_{ab}\,T_{\mathrm{R}}$, where T_{R} is a (representation dependent) normalization. For $SU(N)$ groups, it is conventional to choose $T_{\mathrm{fund}} = \frac{1}{2}$ for the fundamental (defining) representation. Let $A_\mu(x) \equiv A_\mu^a(x)(t_a/i)$ be some gauge field for the gauge group $\mathscr{G}$, with $F_{\mu\nu}(x) \equiv F_{\mu\nu}^a(x)(t_a/i)$ the corresponding field strength.

   (a) Show that components of basis matrices in the adjoint representation are proportional to the structure constants, $(t_b^{\mathrm{adj}})_{ac} \propto f_{ab}{}^c$. What is the precise relation? (Hint: Show that the structure constants satisfy a Jacobi identity.)

(b) If $\psi(x)$ is any field transforming in representation R, then it may be regarded as a dim(R)-dimensional vector with components $\psi^\alpha(x)$, with $\alpha = 1, \cdots, \dim(\mathrm{R})$. What are the corresponding components of $D_\mu \psi(x)$?

(c) For the specific case of a field $\phi(x)$ transforming in the adjoint representation, show that $(D_\mu \phi)^c = \partial_\mu \phi^c + f_{ab}{}^c A^a_\mu \, \phi^b$.

(d) What is $\mathrm{tr}\,(F_{\mu\nu}(x)^2)$ in terms of the components of the field strength?

(e) Let $\Lambda(x) \equiv \Lambda^a(x)(t_a/i)$ denote an infinitesimal gauge transformation. What is the effect of this gauge transformation on: (*i*) the components $A^a_\mu(x)$ of the gauge field, (*ii*) the components $F^a_{\mu\nu}(x)$ of the field strength, (*iii*) the components of some field $\phi(x)$ in the adjoint representation, (*iii*) the components of some field $\psi(x)$ in an arbitrary representation R?

(f) The "quadratic Casimir" C_R is defined by $\sum_a (t^\mathrm{R}_a)^2 = C_\mathrm{R}\, \mathbf{1}$ (where $\mathbf{1}$ is the identity matrix of dimension dim(R)). Explain why $T_\mathrm{R}\, \dim(\mathscr{G}) = \dim(\mathrm{R})\, C_\mathrm{R}$. What is the quadratic Casimir for the fundamental and adjoint representations of $SU(N)$?