1. Gauge fixing. In an arbitrary non-Abelian gauge theory:

   (a) Let $\hat{n}$ be an arbitrary unit vector. Show that any gauge field configuration may be transformed to "axial gauge" in which $\hat{n} \cdot A = 0$. What is the required gauge transformation? Does this condition uniquely determine the gauge field? If not, what is the residual gauge freedom (*i.e.*, what gauge transformations preserve this condition)? What is the appropriate Faddeev-Popov determinant for a gauge-fixing term $\prod_x \delta(\hat{n} \cdot A(x))$?

   (b) Show that any gauge field configuration may be gauge transformed to "radial gauge" in which $x \cdot A(x) = 0$. What is the required gauge transformation?

   (c) A gauge field satisfying $\partial \cdot A = 0$ is said to be in Lorentz gauge. What Faddeev-Popov determinant must accompany a Lorentz-gauge gauge fixing term $\prod_x \delta(\partial \cdot A)$? If this functional delta function is represented as the limit of a Gaussian, $\exp(-\frac{1}{2\xi} \int d^4x \, (\partial^\mu A^a_\mu)^2)$, what is the resulting free gauge-field propagator?

   (d) Extra credit: Given some gauge field $A_\mu(x)$ in Lorentz gauge, apply some infinitesimal gauge transformation $\Lambda(x)$. Derive the condition on Λ such that the result *also* satisfies the Lorentz gauge condition. With reasonable boundary conditions on $\Lambda(x)$ at infinity, can you show that non-zero solutions to this equation do, or do not, exist? Does the conclusion depend on whether the gauge group is Abelian or non-Abelian?

2. Euclidean space propagators.

   (a) Consider the free, massless scalar propagator $\Delta^{(0)}(x-y)$ which is the Green's function for $-\partial^2$. What is its explicit form (as a function of $x-y$, not a Fourier integral) in d-dimensions?

   (b) Consider the free, massive scalar propagator $\Delta^{(0)}(x-y;m)$ which is the Green's function for $-\partial^2 + m^2$. What is its asymptotic behavior for $|x-y| \gg m^{-1}$? (This can be deduced directly from the differential equation. Or you can analyze its Fourier representation.)

   (c) Consider the free, massive propagator $\Delta(x, y; m)$ for a scalar field which transforms in representation R of a non-Abelian gauge group, in the presence of an arbitrary background gauge field $A_\mu(x)$. The propagator is the Green's function for $-D[A]^2 + m^2$, where the covariant derivative is in representation R. Show that for large distances, $|x-y| \gg m^{-1}$, $\Delta(x, y; m) \sim U(x, y)\, \Delta^{(0)}(x-y; m)$, where $U(x, y)$ is the path-ordered exponential of the line integral of the gauge field (in representation R) along the straight line path from y to x.

   (d) Generalize the above results from scalar to fermion propagators.

3. Consider $SU(2)$ Yang-Mills theory, in four-dimensional Euclidean space. Define the "topological charge" $Q \equiv -\frac{1}{16\pi^2} \int d^4x \; \mathrm{tr}\, F_{\mu\nu} \tilde{F}^{\mu\nu}$, where $\tilde{F}^{\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$. Consider field configurations for which $F_{\mu\nu}$ vanishes at infinity faster than $1/r^2$, so that the action $\propto \int d^4x \; \mathrm{tr}\, F^2$ is finite.

   (a) Explain why $\mathrm{tr}\, F_{\mu\nu} \tilde{F}^{\mu\nu}$ is a pseudo-scalar, not a scalar.

   (b) Show that the value of Q is unchanged by any infinitesimal variation of the gauge field $\delta A_\mu(x)$ which vanishes at infinity at least as fast as $1/r$.

   (c) Show that $-\frac{1}{16\pi^2} \mathrm{tr}\, F_{\mu\nu} \tilde{F}^{\mu\nu} = \partial_\mu j^\mu$ for some (pseudo) vector current j^μ. Hint: what pseudo-vectors can be constructed from the gauge field and its derivatives with the correct dimension? Try the most general linear combination and find one that works.

   (d) Explain why the vector potential must approach a "pure gauge", $A_\mu(x) \to g(x) \partial_\mu g(x)^{-1}$, as $|x| \to \infty$, for some gauge transformation $g(x) \in SU(2)$.

(e) Write the topological charge as a surface integral of $j^\mu(x)$, insert the asymptotic pure-gauge form of A_μ, and express the topological charge as a surface integral over the sphere at infinity involving (derivatives of) the gauge transformation $g(x)$. Assume that $g(x)$ approaches some element of the gauge group $g(\hat{x})$, depending only on direction, as $x \to \infty$.

(f) If $g(x)$ approaches a constant group element g_0 as $x \to \infty$, independent of direction, what is the value of the topological charge?

(g) Suppose $g(x) \to i\bar{\tau}_\mu x^\mu/|x|$ as $x \to \infty$, where $\bar{\underset{\sim}{\tau}}$ are the usual Pauli matrices and $\bar{\tau}_0 \equiv -i$. Show that this is a valid $SU(2)$ group element. What is the corresponding value of the topological charge?

(h) What can you infer from these results about the topological classification of finite action gauge configurations?

(i) Suppose a term proportional to the topological charge is added to the Euclidean action defining the functional integral representation of this theory, so that $S_E[A_\mu] = \left[\int d^4x \, \frac{1}{4g^2} F^2\right] + i\,\theta\,Q$. Are there symmetries of the theory at $\theta = 0$ which are not symmetries for non-zero values of θ? Is $\theta = \pi$ special?

(j) What are the equations of motion for this theory? Do they depend on θ? Will any results of perturbation theory depend on θ? Will the partition function $Z = \int \mathcal{D}A_\mu \, e^{-S_E[A]}$, or the corresponding vacuum energy, depend on θ? If so, what can you say about the form of the θ dependence?