1. Dirac Fermions.

   Let $S(x)_{\alpha\beta} \equiv \langle 0|\mathscr{T}\left(\psi_\alpha(x)\bar\psi_\beta(0)\right)|0\rangle = \theta(x^0)\langle 0|\psi_\alpha(x)\bar\psi_\beta(0)|0\rangle - \theta(-x^0)\langle 0|\bar\psi_\beta(0)\psi_\alpha(x)|0\rangle$. Apply to $S(x)$ the operator $i(\gamma^\mu\partial_\mu + m)$. Since $i(\gamma^\mu\partial_\mu + m)\,\psi(x) = 0$, the only reason the result is non-zero is due to the time-ordering, which introduces step functions in time. Differentiating these produces delta functions in time, so that

$$\begin{aligned} i(\gamma^\mu\partial_\mu + m)_{\alpha\beta}\, S(x)_{\beta\gamma} &= (i\gamma^0)_{\alpha\beta}\left[\delta(x^0)\langle 0|\psi_\beta(x)\bar\psi_\gamma(0)|0\rangle + \delta(x^0)\langle 0|\bar\psi_\gamma(0)\psi_\beta(x)|0\rangle\right] \\ &= (i\gamma^0)_{\alpha\beta}\,\langle 0|\{\psi_\beta(\underset{\sim}{x}),\bar\psi_\gamma(0)\}|0\rangle\,\delta(x^0) \\ &= (i\gamma^0)_{\alpha\beta}\,(i\gamma^0)_{\beta\gamma}\,\delta^3(\underset{\sim}{x})\,\delta(x^0) \\ &= \delta_{\alpha\gamma}\,\delta^4(x)\,. \end{aligned}$$

   (Because of the temporal delta function, the anti-commutator of $\psi(x)$ and $\bar\psi(0)$ is an equal-time anti-commutator.) Hence, the propagator $S(x)$ is a Green's function for $i(\gamma^\mu\partial_\mu + m)$.

   Inserting a four-dimensional Fourier representation $S(x) = \int d^4p/(2\pi)^4\, e^{ip\cdot x}\,\tilde S(p)$ into the Green's function equation shows that $i(i\not p + m)\,\tilde S(p) = 1$ or $\tilde S(p) = -i(i\not p + m)^{-1}$. This is a matrix inverse, but it is easy to do if one notices that $(i\not p + m)(-i\not p + m) = (\not p)^2 + m^2 = p^2 + m^2$. Therefore $i\,\tilde S(p) = (-i\not p + m)/(p^2 + m^2)$. There is just one problem: the denominator has poles on the real p^0 axis at $p^0 = \pm\sqrt{\underset{\sim}{p}^2 + m^2}$. These poles are on the integration contour for the temporal Fourier transform, and hence the Fourier integral $S(x) = \int d^4p/(2\pi)^4\, e^{ip\cdot x}\,\tilde S(p)$ is ill-defined. This always happens for Green's functions of wave operators. Specifying a prescription for integrating around the poles in the Fourier representation is equivalent to selecting the Green's function which satisfies appropriate boundary conditions at infinity. In this case, the appropriate prescription is to replace $m^2 \to m^2 - i\epsilon$ in the denominator (just as for a free scalar field). One may justify this by inserting mode expansions and explicitly computing $S(x)$ in the free Dirac theory (and then Fourier transforming the result). Alternatively, one can use the fact that in any quantum theory (with a ground state), time-ordered correlation functions must have analytic continuations to Euclidean space, which implies that $\tilde S(p)$ must be analytic in the first and third quadrants of the complex p^0 plane.

2. Wick's theorem for fermions.

   (a) We have $G_0(x,y) \equiv \Theta(x^0{-}y^0)\,\langle 0|\psi(x)\psi^\dagger(y)|0\rangle - \Theta(y^0{-}x^0)\,\langle 0|\psi^\dagger(y)\psi(x)|0\rangle$. Differentiate with respect to time, recalling that $\partial_t\psi(x) \equiv i[\hat H, \psi(x)]$:

$$\begin{aligned} \frac{\partial}{\partial x^0}\,G_0(x,y) &= i\Theta(x^0{-}y^0)\,\langle 0|[\hat H,\psi(x)]\psi^\dagger(y)|0\rangle + \delta(x^0{-}y^0)\,\langle 0|\psi(x)\psi^\dagger(y)|0\rangle \\ &\qquad - i\Theta(y^0{-}x^0)\,\langle 0|\psi^\dagger(y)[\hat H,\psi(x)]|0\rangle + \delta(y^0{-}x^0)\,\langle 0|\psi^\dagger(y)\psi(x)|0\rangle \\ &= i\langle 0|\mathscr{T}\left([\hat H,\psi(x)]\,\psi^\dagger(y)\right)|0\rangle + \delta(x^0{-}y^0)\,\langle 0|\{\psi(x),\psi^\dagger(y)\}|0\rangle\,. \end{aligned}$$

   For the free Hamiltonian $\hat H = \int d^3x\,\psi^\dagger(x)\,h\,\psi(x)$ (with h some differential operator in x), the commutator is simple, $[\hat H,\psi(x)] = -\int d^3x\,\{\psi^\dagger(z),\psi(x)\}\,h\,\psi(z) = -h\,\psi(x)$. Thus

$$i\frac{\partial}{\partial x^0}\,G_0(x,y) = h\,\langle 0|\mathscr{T}\left(\psi(x)\psi^\dagger(y)\right)|0\rangle + i\delta(x^0{-}y^0)\,\delta^3(\underset{\sim}{x}{-}\underset{\sim}{y})\,,$$

   or $(i\partial_0 - h)\,G_0(x,y) = i\,\delta^4(x{-}y)$ (with h understood as acting on x dependence).

   (b) The linear operator $\mathcal{K} \equiv i\partial_0 - h$ is invertible if (and only if) there are no zero mode solutions $f(x)$ satisfying $\mathcal{K}\,f(x) = 0$ *within* the appropriate space of functions on which $\mathcal{K}$ acts. One way to check this is to diagonalize $\mathcal{K}$ and see if any eigenvalues vanish. In the case at hand, this is easy since the operator $i\partial_0 - h$ is time-translation invariant and the sum of two mutually commuting terms. Hence its eigenvectors are products of plane waves in time, $e^{-i\omega t}$, multiplied by eigenfunctions of h,

$$(i\partial_0 - h)\, e^{-i\omega x^0}\, \chi_n(\underset{\sim}{x}) = (\omega - \epsilon_n)\, e^{-i\omega x^0}\, \chi_n(\underset{\sim}{x})\,.$$

Here $\{\chi_n(\underset{\sim}{x})\}$ denote the orthonormal eigenfunctions of h with corresponding eigenvalues $\{\epsilon_n\}$. Because the Hamiltonian is hermitian, the single particle energies ϵ_n must be real.

The eigenvalue $\omega - \epsilon_n$ *will* vanish if ω can equal ϵ_n. However, we are considering the invertibility of this operator when the time x^0 is analytically continued through a phase $e^{-i\theta}$ (with $0 < \theta < \pi$). This means that the space of functions on which the time derivative acts should be functions which are well behaved on the line $t = e^{-i\theta}\, s$ (for s real). The plane wave $e^{-i\omega t}$ blows up exponentially as $|t| \to \infty$ in one direction or the other along this line *unless* the frequency ω lies on the straight line C which is rotated counterclockwise from the real axis by the angle θ. Hence, the only well-behaved eigenfunctions are those for which the frequency ω lies on the line C. So (as long as the single particle Hamiltonian h has no zero eigenvalues), the frequency ω cannot equal a single-particle energy ϵ_n, and so $i\partial_0 - h$ is invertible. The explicit form of the inverse is easily constructed by projecting into the basis of eigenfunctions:

$$\begin{aligned}
G_0(x,y) &= i\,\langle x|(i\partial_0 - h)^{-1}|y\rangle \\
&= i \int_C \frac{d\omega}{2\pi} \sum_n e^{-i\omega x^0}\, \chi_n(\underset{\sim}{x})\, \frac{1}{\omega - \epsilon_n}\, \chi_n^\dagger(\underset{\sim}{y})\, e^{i\omega y^0} \\
&= \sum_n \left[\Theta(x^0 - y^0)\,\Theta(\epsilon_n) - \Theta(y^0 - x^0)\,\Theta(-\epsilon_n)\right] e^{-i\epsilon_n(x^0 - y^0)}\, \chi_n(\underset{\sim}{x})\chi_n^\dagger(\underset{\sim}{y})\,.
\end{aligned}$$

Here, the frequency integral is computed as usual by closing the contour downward for $x^0 > y^0$, and upward otherwise, picking up the residue of the simple pole at $\omega = \epsilon_n$ if it lies within the resulting closed contour. This is precisely the same expression which results from inserting the mode expansion of the fermion field,

$$\hat{\psi}(x) = \sum_n \hat{c}_n\, \chi_n(\underset{\sim}{x})\, e^{-i\epsilon_n t}$$

into the definition of the propagator, and evaluating the matrix elements using the fact that the ground state $|0\rangle$ satisfies $\hat{c}_n|0\rangle = 0$ if $\epsilon_n > 0$, and $\hat{c}_n^\dagger|0\rangle = 0$ if $\epsilon_n < 0$.

(c) To find the equation of motion of the higher correlation functions,

$$G_0^{(n,m)}(x_1 \ldots x_n; y_1 \ldots y_m) \equiv \langle 0|\mathscr{T}\left(\psi(x_1)\cdots\psi(x_n)\psi^\dagger(y_m)\cdots\psi^\dagger(y_1)\right)|0\rangle\,,$$

one may begin by differentiating with respect to x_1^0 (or any other time variable), just as in (a). A slightly more clever approach is to apply the combination $i\,(\partial/\partial x_1^0) - h$ (where h is regarded as acting on functions of $\underset{\sim}{x}_1$) since $\left[i\,(\partial/\partial x_1^0) - h\right]\psi(x_1) = 0$ is precisely the Heisenberg equation of motion for $\psi(x)$. Thus if the time-ordering symbol were not present, this operator applied to $G_0^{(n,m)}$ would simply give zero. But, because of the step functions introduced by the time-ordering symbol, as the time x_1^0 varies there is a discontinuity every time x_1^0 crosses one of the other times. Differentiating these step functions will give a delta function of a time difference, just as in (a). Consequently,

$$\left(i\frac{\partial}{\partial x_1^0} - h\right) G_0^{(n,m)}(x_1\cdots x_n; y_1\cdots y_m)$$

$$= i\sum_{j=1}^{m} (-)^{n-1+m-j}\,\delta(x_1^0 - y_j^0)$$

$$\times \langle 0|\mathscr{T}\left(\{\psi(x_1),\psi^\dagger(y_j)\} + \psi(x_2)\cdots\psi(x_n)\psi^\dagger(y_m)\cdots\psi(y_{j+1})\psi(y_{j-1})\cdots\psi(y_1)\right)|0\rangle$$

$$= i\sum_{j=1}^{m} (-)^{n-1+m-j}\,\delta^4(x_1 - y_j)\, G_0^{(n-1,m-1)}(x_2\cdots x_n; y_m\cdots \not{y}_j\cdots y_1)\,.$$

The sign of each term depends on the number of minus signs introduced by the time-ordering symbol interchanging fermionic operators; $(-)^{n-1+m-j}$ is number of interchanges needed to bring the operator $\psi^\dagger(y_j)$ next to $\psi(x_1)$ starting from its original position in the product. Note that no delta function terms result when the time x_1^0 crosses the time of one of the other $\{\psi(x_k)\}$ since that discontinuity is proportional to $\{\psi(x_1),\psi(x_k)\} = 0$.

This result is a linear equation for $G_0^{(n,m)}$ with the same operator on the left-hand side discussed above. If all times are analytically continued from the real axis to the rotated line with phase $-\theta$, then the discussion in (b) may be repeated verbatim. Since the operator $i\partial_t - h$ is invertible (when acting on functions which are well-behaved on the rotated line), the equation must have a unique solution in this function space.

(d) The equation above is "solved" by applying the inverse of $(i\partial_t - h)$ to both sides,

$$G_0^{(n,m)}(x_1\cdots x_n; y_1\cdots y_m) = i\sum_{j=1}^{m} (-)^{n-1+m-j}\,\langle x_1|\left(i\frac{\partial}{\partial x_1^0} - h\right)^{-1}|y_j\rangle$$

$$\times G_0^{(n-1,m-1)}(x_2\cdots x_n; y_m\cdots \not{y}_j\cdots y_1)$$

$$= \sum_{j=1}^{m} (-)^{n-1+m-j}\, G_0(x_1,y_j)\, G_0^{(n-1,m-1)}(x_2\cdots x_n; y_m\cdots \not{y}_j\cdots y_1)\,.$$

The same relation may be used to reexpress the factors of $G_0^{(n-1,m-1)}$ on the right hand side, yielding a new identity for $G_0^{(n,m)}$ as a sum of $m(m-1)$ terms, each of which is a product of two propagators times $G_0^{(n-2,m-2)}$. Or, iterating once more, $m(m-1)(m-2)$ terms involving three propagators times $G_0^{(n-3,m-3)}$, etc. If $n \neq m$, the process terminates when there are either no more ψ's or $\psi^\dagger$'s in the resulting correlation function; in other words every term in the sum has a factor of $G_0^{(n-m,0)}$ (or $G_0^{(0,m-n)}$) which vanishes. If $n = m$, the iteration leads to a sum of $n!$ terms, each of which involves a product of n propagators (and a particular pairing of the $\{x_i\}$ and $\{y_j\}$ coordinates),

$$G_0^{(n,m)} = \delta_{nm}\sum_{\pi} \sigma_\pi\, G_0(x_1,y_{\pi_1})\, G_0(x_2,y_{\pi_2})\cdots G_0(x_n,y_{\pi_n})\,.$$

This is Wick's theorem for any theory of fermions. The sum extends over all $n!$ permutations of the $\{y_j\}$ coordinates. To check that the resulting sign equals the signature σ_π of the permutation (-1 for odd permutations and $+1$ for even ones), note that (when $n = m$) the factor of $(-)^{n+m+i-1} = (-)^{i-1}$ in the first result is the same as the number of transpositions which are needed to cyclically permute $y_1 y_2\cdots y_j$ to $y_j y_1 y_2\cdots y_{j-1}$ (leaving the other coordinates unchanged). So, at every stage in the iteration, the overall sign is just the signature of the permutation which has rearranged the $\{y_j\}$.

(e) At non-zero temperature, one is interested not in ground state expectation values, but rather thermal correlation functions,

$$G_0^{(n,m)}(x_1 \ldots x_n; y_1 \ldots y_m) = \mathrm{Tr}\,\left[\rho\,\mathscr{T}\left(\psi(x_1)\cdots\psi(x_n)\psi^\dagger(y_m)\cdots\psi^\dagger(y_1)\right)\right],$$

where $\rho \equiv e^{-\beta\hat{H}}/Z$ is the density matrix describing the canonical equilibrium thermal distribution at temperature $kT = 1/\beta$. "Turning on" a non-zero temperature does not change *in any way* the canonical commutation relations, the Hamiltonian, the quantum time-evolution, or the time-ordered product of operators $\mathscr{T}\left(\psi(x_1)\cdots\psi(x_n)\psi^\dagger(y_m)\cdots\psi^\dagger(y_1)\right)$. Since the only ingredients used in deriving the equation of motion for $G_0^{(n,m)}$ were the definition of time-ordering, the Heisenberg equation of motion for $\psi(x)$, and the canonical commutation relations, the finite temperature correlation functions satisfy exactly the same equations of motion as at zero temperature. What does change is the boundary conditions. If one analytically continues all times to the imaginary axis, a short exercise shows that time-ordered thermal correlation functions satisfy

$$G(x^0{-}y^0, \underset{\sim}{x}, \underset{\sim}{y}) = \pm G(x^0{-}y^0{-}i\beta, \underset{\sim}{x}, \underset{\sim}{y}),$$

if $x^0{-}y^0$ lies within the strip $[0, i\beta]$. This means that thermal correlation functions are (analytic continuations of) functions which are periodic (for bosonic operators), or antiperiodic (for fermionic operators) in imaginary time with period β. This is known as Euclidean periodicity (or KMS boundary conditions, after Kubo, Martin and Schwinger). The same argument used above shows that the linear operator $i\partial_t - h$ is invertible when acting on functions obeying the thermal boundary condition, since the periodicity condition forces the frequency ω of allowed eigenfunctions to have quantized imaginary values, $\omega_k \equiv (2k{+}1)i\pi/\beta$ for fermions or $\omega_k \equiv 2ki\pi/\beta$ for bosons. Every step in the proof of Wick's theorem works the same as at zero temperature, only the explicit form of the propagator changes and becomes

$$G_0(x,y) = i\,\langle x|(i\partial_0 - h)^{-1}|y\rangle = i\beta^{-1}\sum_k\sum_n e^{-i\omega_k x^0}\,\chi_n(\underset{\sim}{x})\,\frac{1}{\omega_k - \epsilon_n}\,\chi_n^\dagger(\underset{\sim}{y})\,e^{i\omega_k y^0}.$$

(One can perform the sum over the discrete frequencies explicitly.) The fact that Wick's theorem holds without change at non-zero temperature means that the entire structure of diagrammatic perturbation theory has an immediate generalization to finite temperature calculations. See books like Rickayzen for lots of interesting applications of finite temperature perturbation theory.

3. Gamma Matrix Identities.

(a) For any μ, γ^μ anticommutes with three out of the four matrices in $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, and hence anticommutes with γ_5. And $(\gamma_5)^2 = -(\gamma^0\gamma^1\gamma^2\gamma^3)(\gamma^0\gamma^1\gamma^2\gamma^3) = +(\gamma^0\gamma^1\gamma^2)(\gamma^0\gamma^1\gamma^2)(\gamma^3)^2 = +(\gamma^0\gamma^1)(\gamma^0\gamma^1)(\gamma^2)^2(\gamma^3)^2 = -(\gamma^0)^2(\gamma^1)^2(\gamma^2)^2(\gamma^3)^2 = -(-1)(+1)^3 = 1$.

(b) Insert $1 = (\gamma_5)^2$ into the front of the trace, move one γ_5 past all the other gamma matrices to the end of the trace, using its anticommutation relation, and then use trace cyclicity to bring it back to the front: $\mathrm{tr}\,\gamma_{\mu_1}\cdots\gamma_{\mu_k} = \mathrm{tr}\,(\gamma_5)^2\gamma_{\mu_1}\cdots\gamma_{\mu_k} = (-1)^k\,\mathrm{tr}\,\gamma_5\gamma_{\mu_1}\cdots\gamma_{\mu_k}\gamma_5 = (-1)^k\,\mathrm{tr}\,(\gamma_5)^2\gamma_{\mu_1}\cdots\gamma_{\mu_k} = (-1)^k\,\mathrm{tr}\,\gamma_{\mu_1}\cdots\gamma_{\mu_k}$. If k is odd, then the trace equals minus itself and hence equals zero.

(c) $\mathrm{tr}\,\not{a}\not{b} = a_\mu b_\nu\,\mathrm{tr}\,\gamma^\mu\gamma^\nu = \frac{1}{2}\,a_\mu b_\nu\,\mathrm{tr}\,(\{\gamma^\mu,\gamma^\nu\}) = a_\mu b_\nu\,g^{\mu\nu}\,\mathrm{tr}\,1 = 4\,a\cdot b$. For $\mathrm{tr}\,\not{a}\not{b}\not{c}\not{d}$, first move the $\not{a}$ from the beginning to the end, using $\not{a}\not{b} = -\not{b}\not{a} + 2a\cdot b$, *etc.*, and then use trace cyclicity to bring the final $\not{a}$ back to the front:

$$\begin{aligned}\mathrm{tr}\ \not{a}\not{b}\not{c}\not{d} = 2(a\cdot b)\mathrm{tr}\ \not{c}\not{d} - \mathrm{tr}\ \not{b}\not{a}\not{c}\not{d} &= 2(a\cdot b)\,\mathrm{tr}\ \not{c}\not{d} - 2(a\cdot c)\,\mathrm{tr}\ \not{b}\not{d} + 2(a\cdot d)\,\mathrm{tr}\ \not{b}\not{c} - \mathrm{tr}\ \not{b}\not{c}\not{d}\not{a} \\ &= 8(a\cdot b)(c\cdot d) - 8(a\cdot c)(b\cdot d) + 8(a\cdot d)(b\cdot c) - \mathrm{tr}\ \not{a}\not{b}\not{c}\not{d}\,.\end{aligned}$$

Hence $\mathrm{tr}\ \not{a}\not{b}\not{c}\not{d} = 4\,[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)]$.

(d) Let N_k denote the number of terms in a trace of $2k$ gamma matrices. The same strategy used in part (e) shows that $N_k = (2k-1)\,N_{k-2}$. Since $N_1 = 1$, the result is $N_k = (2k-1)(2k-3)\cdots(3)(1) = (2k-1)!!$, which grows like $(k/e)^{k/2}$ for k large.

(e) (i) $\gamma^\mu\gamma_\mu = g_{\mu\nu}\,\gamma^\mu\gamma_\nu = \frac{1}{2}\,g_{\mu\nu}\,\{\gamma^\mu,\gamma_\nu\} = g_{\mu\nu}\,g^{\mu\nu} = 4$ (times a 4×4 unit matrix).
(ii) $\gamma^\mu\not{a}\gamma_\mu = a_\nu\,\gamma^\mu\gamma^\nu\gamma_\mu = a_\nu\,(2g^{\mu\nu} - \gamma^\nu\gamma^\mu)\gamma_\mu = 2a^\mu\gamma_\mu - 4a_\nu\gamma^\nu = -2\not{a}$.
(iii) $\gamma^\mu\not{a}\not{b}\gamma_\mu = (2a^\mu - \not{a}\gamma^\mu)\not{b}\gamma_\mu = 2\not{b}\not{a} - \not{a}(2b^\mu - \not{b}\gamma^\mu)\gamma_\mu = 2\not{b}\not{a} - 2\not{a}\not{b} + 4\not{a}\not{b} = 2\{\not{b},\not{a}\} = 4\,a\cdot b$.
(iv) $\gamma^\mu\not{a}\not{b}\not{c}\gamma_\mu = (2a^\mu - \not{a}\gamma^\mu)\not{b}\not{c}\gamma_\mu = 2\not{b}\not{c}\not{a} - \not{a}(2b^\mu - \not{b}\gamma^\mu)\not{c}\gamma_\mu = 2\not{b}\not{c}\not{a} - 2\not{a}\not{c}\not{b} - 2\not{a}\not{b}\not{c} = 2\not{b}\not{c}\not{a} - 4\not{a}(b\cdot c) = -2\not{c}\not{b}\not{a}$.

4. Majorana representation. If $\psi'(x) = S\,\psi(x)$ for some unitary matrix S, then $(\psi'(x)^\dagger)^T = S^*(\psi^\dagger(x))^T = S^*C\,\psi(x) = S^*C\,S^{-1}\psi'(x)$. So the problem reduces to finding a unitary matrix S such that $S^*C\,S^{-1} = 1$ or $C = S^TS$. In our standard representation, $C = \begin{bmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{bmatrix}$. It has two eigenvalues of $+1$, with corresponding eigenvectors $(0,1,1,0)^T$ and $(1,0,0,-1)^T$ (up to arbitrary phases), and two eigenvalues of -1 with eigenvectors $(1,0,0,1)^T$ and $(0,1,-1,0)^T$ (up to phases). Normalize these eigenvectors and stack them together to form a unitary matrix U which diagonalizes C, so that $C = U\lambda U^\dagger$ with λ diagonal. The matrix $S = U^\dagger$ will be a solution to $C = S^TS$ if we can select phases so that $U\lambda = (U^\dagger)^T = U^*$. To satisfy this, choose the eigenvectors with eigenvalue -1 to be pure imaginary, and while the eigenvectors with eigenvalue $+1$ are real. One explicit solution (which can be written compactly) is $S = \frac{1}{\sqrt{2}}\begin{bmatrix} -i & \sigma_2 \\ \sigma_1 & \sigma_3 \end{bmatrix}$. This change of basis induces a similarity transformation on the γ-matrices, $\tilde{\gamma}_\mu \equiv S\gamma_\mu S^\dagger$ with the transformed matrices $\tilde{\gamma}_\mu$ all being real:

$$\tilde{\gamma}_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \tilde{\gamma}_1 = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}, \qquad \tilde{\gamma}_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \tilde{\gamma}_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$