1. Free particle spinors.

(a) Given the explicit forms introduced in lecture, \( \alpha \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) and \( \beta \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \), verifying their anticommutations relations is immediate, \( \{ \alpha_i, \alpha_j \} = 2 \delta_{ij}, \{ \alpha_i, \beta \} = 0, \) and \( \{ \beta, \beta \} = 2 \beta^2 = 2. \) Defining \( \gamma_0 = i \beta \) and \( \gamma_i = -i \beta \alpha_i, \) one easily verifies that \( \{ \gamma_i, \gamma_j \} = \{ \alpha_i, \alpha_j \} = 2 \delta_{ij}, \{ \gamma_0, \gamma_0 \} = \alpha_i - \alpha_i = 0, \) and \( \{ \gamma_0, \gamma_0 \} = (\gamma_0)^2 = -2, \) or equivalently \( \{ \gamma_\mu, \gamma_\nu \} = 2 \eta_{\mu\nu} \) with \( \eta_{\mu\nu} \equiv \text{diag}(-1, 1, 1, 1) \) the Minkowski space metric tensor.

(b) The spinors \( u_\pm(p) \) satisfy the the eigenvalue equation \( h(p) u_\pm(p) = p^0 u_\pm(p), \) with \( p^0 \equiv E(p) \), while the spinors \( v_\pm(-p) \) satisfy \( h(p) v_\pm(-p) = -p^0 v_\pm(-p). \) Note that \( h(p) \equiv \alpha \cdot p + \beta m = i \gamma^0(\gamma \cdot p + m), \) since \( \beta = i \gamma^0 \) and \( \alpha = \gamma \gamma^7. \) Multiplying both sides of the eigenvalue equation for \( u_\pm(p) \) (from the left) by \( i \gamma^0 \) gives \( (i \gamma \cdot p + m) u_\pm(p) = i \gamma^0 p^0 u_\pm(p) \) which is the same as \( (i \gamma^\mu p_\mu + m) u_\pm(p) = 0. \) Doing the same for \( v_\pm(-p), \) and then changing the dummy variable \( p \rightarrow -p \) gives \( (i \gamma \cdot p + m) v_\pm(p) = -i \gamma^0 p^0 v_\pm(p), \) which is the same as \( (-i \gamma^\mu p_\mu + m) v_\pm(p) = 0. \)

(c) Since \( (\gamma^0)^\dagger = -\gamma^0 \) and \( \gamma^\dagger = \gamma, \) we have \( (\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0 \) (because \( \gamma^0 \) anticommutes with \( \gamma \) and squares to \( -1 \)). Take the Hermitian conjugate of \( (i \gamma^\mu p_\mu + m) u_\pm(p) = 0 \) and insert this relation to get \( u_\pm(p)(-i \gamma^0 \gamma^\mu \gamma^0 p_\mu + m) = 0 \) or \( u_\pm(p)(i \gamma^0 (i \gamma^\mu p_\mu + m)(i \gamma^0)) = 0. \) Multiplying (on the right) by \( i \gamma^0 \) and defining \( \bar{u}_\pm(p) = u_\pm(p)^\dagger (i \gamma^0) \) gives the result \( \bar{u}_\pm(p)(i \gamma^\mu p_\mu + m) = 0. \) Exactly the same steps applied to \( (-i \gamma^\mu p_\mu + m) v_\pm(p) = 0 \) yield \( \bar{v}_\pm(p)(-i \gamma^\mu p_\mu + m) = 0. \)

(d) Write zero in a non-trivial way, using the results of parts (a) and (b):
\[
0 = \bar{u}_s(p) (i \gamma^0)(i \gamma^\mu p_\mu + m) s_{s'}(p) + \bar{u}_s(p)(i \gamma^\mu p_\mu + m)(i \gamma^0) s_{s'}(p).
\]
Collecting terms gives \( 0 = 2m \bar{u}_s(p)(i \gamma^0) u_{s'}(p) - p_{s} \bar{u}_s(p) \{ \gamma^0, \gamma^\mu \} u_{s'}(p) \) and using the gamma matrix algebra reduces this to \( 2m u_{s'}(p)^\dagger u_s(p) = 2E(p) u_{s'}(p). \) Since \( u_{s'}(p)^\dagger u_s(p) = 2E(p) \delta_{ss'}, \) this shows that \( \bar{u}_s(p) u_{s'}(p) = 2m \delta_{ss'}. \) Applying exactly the same steps to \( 0 = \bar{v}_s(p)(i \gamma^0)(-i \gamma^\mu p_\mu + m) v_{s'}(p) + \bar{v}_s(p)(-i \gamma^\mu p_\mu + m)(i \gamma^0) v_{s'}(p) \) gives \( \bar{v}_s(p) v_{s'}(p) = -2m \delta_{ss'}. \) And starting from \( 0 = \bar{v}_s(p)(i \gamma^0)(i \gamma^\mu p_\mu + m) u_{s'}(p) - \bar{v}_s(p)(-i \gamma^\mu p_\mu + m)(i \gamma^0) u_{s'}(p) \) or \( 0 = \bar{u}_s(p)(i \gamma^0)(-i \gamma^\mu p_\mu + m) v_{s'}(p) - \bar{u}_s(p)(i \gamma^\mu p_\mu + m)(i \gamma^0) v_{s'}(p) \) gives \( \bar{v}_s(p) u_{s'}(p) = \bar{u}_s(p) v_{s'}(p) = 0. \)

(e) Since \( h(p) \) is Hermitian, its eigenvectors are orthogonal. Normalizing them to \( 2E(p) \) means that the corresponding completeness relation is \( \sum_s u_s(p) u_s(p)^\dagger + v_s(-p) v_s(-p)^\dagger = 2E(p) \) (where a \( 4 \times 4 \) identity matrix is implicit on the RHS). And the representation of \( h(p) \) in terms of projections onto its eigenvectors is \( \sum_s u_s(p) u_s(p)^\dagger - v_s(-p) v_s(-p)^\dagger = 2h(p). \) Adding and subtracting these two relations gives \( \sum_s u_s(p) u_s(p)^\dagger = E(p) + h(p) \) and \( \sum_s v_s(-p) v_s(-p)^\dagger = E(p) - h(p). \) Multiply (on the right) by \( i \gamma^0 \) to convert these to
\[
\Lambda_+(p) = \sum_s u_s(p) \bar{u}_s(p) = (E(p) + h(p))(i \gamma^0) = [p^0 + i \gamma^0 (i \gamma \cdot p + m)](i \gamma^0) = -i \gamma^\mu p_\mu + m,
\]
\[
\Lambda_-(p) = -\sum_s v_s(p) \bar{v}_s(p) = -(E(p) + h(p))(i \gamma^0) = [-p^0 + i \gamma^0 (-i \gamma \cdot p + m)](i \gamma^0) = i \gamma^\mu p_\mu + m.
\]

(f) Note that \( (\hat{p})^2 = (\gamma^\mu a_\mu)^2 = \frac{1}{4} (\gamma^\mu, \gamma^\nu) a_\mu a_\nu = g^{\mu\nu} a_\mu a_\nu = a^2, \) for any 4-vector \( a. \) Hence, \( \Lambda_+(p)^2 = (\mp i \hat{p} + m)^2 = -(\hat{p})^2 + 2m \hat{p} + m^2 = \mp 2m \hat{p} + m^2 - p^2 = 2m(\mp i \hat{p} + m) = 2m \Lambda_+(p), \) where the fact that \( p \) is an on-shell momentum satisfying \( p^2 + m^2 = 0 \) was used. Similarly, \( \Lambda_+(p) \Lambda_+(p) = (\mp i \hat{p} + m)(\pm i \hat{p} + m) = (\hat{p})^2 + m^2 = p^2 + m^2 = 0 \). And \( \Lambda_-(p) + \Lambda_-(p) = 2m. \) Since \( \Lambda_+(p)/2m \) equals its square, its eigenvalues can only be zero or one, and hence each matrix projects onto the subspace spanned by its eigenvectors with eigenvalue one. Since \( \Lambda_+(p) \Lambda_+(p) = 0, \) these subspaces are mutually orthogonal.
(g) Let \( \chi_{\pm}(p) \) denote two-component spinors of definite helicity, so \((\sigma \cdot \hat{p}) \chi_{\pm}(p) = \pm \chi_{\pm}(p)\). Define \(\eta(p) = -i\sigma_2 \chi_{\pm}(p)\), and evaluate this spinor’s helicity: \((\sigma \cdot \hat{p}) \eta(p) = [(\sigma \cdot \hat{p}) (-i\sigma_2) \chi_{\pm}(p)]^* = [(i\sigma_2) (\sigma \cdot \hat{p}) \chi_{\mp}(p)]^* = (-i\sigma_2) \chi_{\mp}(p) = \eta(p)\). [This used the fact that \(i\sigma_2\) is real, and that \(\sigma^2 \sigma_2 = -\sigma_2 \sigma_2\).] Therefore \(\eta(p)\) is a two-spinor of positive helicity, which means it must equal \(\chi_{\pm}(p)\) up to some phase factor. So \(-i\sigma_2 \chi_{\pm}(p) = e^{i\alpha} \chi_{\mp}(p)\), for some phase \(\alpha\). We never defined the overall phases of \(\chi_{\pm}(p)\), but, given any choice of overall phase for \(\chi_{\mp}(p)\), we may choose the phase of \(\chi_{\pm}(p)\) so that this relation is satisfied with \(\alpha = 0\). Having done so, the complex conjugate of this relation gives \(-i\sigma_2 \chi_{\mp}(p) = \chi_{\pm}(p)\) or (multiplying by \(-i\sigma_2\)) \(-i\sigma_2 \chi_{\pm}(p) = -\chi_{\pm}(p)\), as claimed.

(h) We have \(u_{\pm}(p) = A \begin{pmatrix} (E+m) \chi_{\pm}(p) \\ (\sigma^\tau \cdot \hat{p}) \chi_{\pm}(p) \end{pmatrix}\) and \(v_{\pm}(p) = \pm A \begin{pmatrix} (E+m) \chi_{\pm}(p) \\ (\sigma^\tau \cdot \hat{p}) \chi_{\pm}(p) \end{pmatrix}\) (in the standard representation), where \(A = (E+m)^{-1/2}\). Equivalently, \(v_{\pm}(p) = \pm A \begin{pmatrix} (\sigma^\tau \cdot \hat{p}) \chi_{\pm}(p) \\ (E+m) \chi_{\pm}(p) \end{pmatrix}\). Inserting the result of part (g), namely \(\chi_{\pm}(p) = \mp i\sigma_2 \chi_{\mp}(p)\), into \(u_{\pm}(p)^*\) gives \(u_{\pm}(p)^* = A \begin{pmatrix} \mp i\sigma_2 (E+m) \chi_{\mp}(p) \\ (\sigma^\tau \cdot \hat{p}) \chi_{\mp}(p) \end{pmatrix}\) and \(a_{\pm}(p)^* = A \begin{pmatrix} \mp i\sigma_2 (E+m) \chi_{\mp}(p) \\ (\sigma^\tau \cdot \hat{p}) \chi_{\mp}(p) \end{pmatrix}\).

(i) \(C = \gamma_2 = i\alpha_2 \beta\) anticommutates with \(\beta\) and \(\alpha_2\), and commutes with \(\alpha_1\) and \(\alpha_3\). In the standard representation, \(\beta\), \(\alpha_1\) and \(\alpha_3\) are symmetric, while \(\alpha_2\) is antisymmetric. Hence, \(C^{-1} \beta \gamma C = -\beta\), \(C^{-1} T \alpha_1 C = -\alpha_1\) and \(C^{-1} T \alpha_2 C = -\alpha_2\). Finally, \(C^T = C^{-1} = C\), giving the claimed results.

2. Gamma Matrix Identities.

(a) For any \(\mu\), \(\gamma^\mu\) anticommutates with three out of the four matrices in \(\gamma_5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3\), and hence anticommutates with \(\gamma_5\). And \((\gamma_5)^2 = - (\gamma^0 \gamma^1 \gamma^2 \gamma^3) (\gamma^0 \gamma^1 \gamma^2 \gamma^3) = + (\gamma^0 \gamma^1 \gamma^2) (\gamma^0 \gamma^1 \gamma^2) (\gamma^3)^2 = + (\gamma^0 \gamma^1)^2 (\gamma^2)^2 (\gamma^3)^2 = - (\gamma^0 \gamma^1)^2 (\gamma^2)^2 (\gamma^3)^2 = -(1)(1)^2 = 1\).

(b) Insert \(1 = (\gamma_5)^2\) into the front of the trace, move one \(\gamma_5\) past all the other gamma matrices to the end of the trace, using its anticommutation relation, and then use trace cyclicity to bring it back to the front: \(\text{tr} \gamma_{\mu_1} \cdots \gamma_{\mu_k} = \text{tr} (\gamma_5)^2 \gamma_{\mu_1} \cdots \gamma_{\mu_k} = -(1)^k \text{tr} \gamma_{\mu_1} \cdots \gamma_{\mu_k} \gamma_{\mu_k} = - (1)^k \text{tr} \gamma_{\mu_1} \cdots \gamma_{\mu_k} \gamma_{\mu_k}\).

(c) \(\text{tr} \, \phi^\mu \, \phi^\nu = a_{\mu} b_{\nu} \, \text{tr} \{ \gamma^\mu, \gamma^\nu \} = 1/2 \, a_{\mu} b_{\nu} \, \text{tr} \{ \{ \gamma^\mu, \gamma^\nu \} \} = a_{\mu} b_{\nu} \, g^{\mu\nu} \text{tr} \, 1 = 4 \, a \cdot b\). For \(\phi \phi \phi \phi\), first move the \(\phi\) from the beginning to the end, using \(\phi \phi = -\phi \phi + 2 a \cdot b\), etc., and then use trace cyclicity to bring the final \(\phi\) back to the front:

\[
\text{tr} \, \phi \phi \phi \phi = 2(a \cdot b) \text{tr} \, \phi \phi \phi \phi - \text{tr} \, \phi \phi \phi \phi = 2(a \cdot b) \text{tr} \, \phi \phi + 2(a \cdot d) \text{tr} \, \phi \phi + 2(a \cdot d) \text{tr} \, \phi \phi - \text{tr} \, \phi \phi \phi \phi
= 8(a \cdot b)(c \cdot d) - 8(a \cdot c)(b \cdot d) + 8(a \cdot d)(b \cdot c) - \text{tr} \, \phi \phi \phi \phi.
\]

Hence \(\text{tr} \, \phi \phi \phi \phi = 4 \{ (a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c) \} \).

(d) Let \(N_k\) denote the number of terms in a trace of 2\(k\) gamma matrices. The same strategy used in part (e) shows that \(N_k = (2k-1)(2k-3) \cdots (3)(1) = (2k-1)!!\), which grows like \((k/e)^{k/2}\) for \(k\) large.

(e) (i) \(\gamma^\mu \gamma_\mu = g_{\mu\nu} \gamma^\mu \gamma_\nu = 1/2 g_{\mu\nu} \{ \gamma^\mu, \gamma_\nu \} = g_{\mu\nu} g^{\mu\nu} = 4\) (times a \(4 \times 4\) unit matrix).
(ii) \(\gamma^\mu \phi_\mu = a_\mu \gamma^\mu \gamma_\mu = a_\mu (2g^{\mu\nu} - \gamma^\mu \gamma_\mu = 2a_\mu \gamma_\mu - 4a_\mu \gamma_\mu = -2a_\mu\).
(iii) \(\gamma^\mu \phi_\nu = (2a_\mu - \theta^\mu \gamma_\mu = 2b_\mu - \theta^\mu \gamma_\mu = 2b_\mu - 2b_\mu + 4b_\mu = 4a \cdot b\).
(iv) \(\gamma^\mu \phi_\mu \phi_\mu = (2a^\mu - \phi^\mu \gamma_\mu = 2b^\mu - \phi^\mu \gamma_\mu = 2b^\mu - 2b^\mu - 4b^\mu = 2b^\mu - 4b^\mu = -2b^\mu\).
3. Fermionic Oscillators.

(a) This is a two-level system. The Hamiltonian has an eigenstate $|0\rangle$, defined by $\hat{\psi}|0\rangle = 0$, with energy 0, and another eigenstate $|1\rangle \equiv \hat{\psi}^\dagger|0\rangle$ with energy $\Omega$. If $\Omega > 0$ then $|0\rangle$ is the ground state, while $|1\rangle$ is the excited state. The partition function $Z = 1 + e^{-\beta\Omega}$.

(b) $G^{(2)}(\tau; \tau') = \Theta(\tau - \tau')\langle \hat{\psi}(-i\tau)\hat{\psi}^\dagger(-i\tau') \rangle - \Theta(\tau' - \tau)\langle \hat{\psi}^\dagger(-i\tau')\hat{\psi}(-i\tau) \rangle$, where $\langle \cdots \rangle$ denotes a thermal average, so $\langle \cdots \rangle = Z^{-1}\langle 0|\cdots|0 \rangle + e^{-\beta\Omega}\langle 1|\cdots|1 \rangle$, and $\hat{\psi}(-i\tau) = e^{H\tau}\hat{\psi} e^{-H\tau} = e^{-i\tau}\hat{\psi}$. Putting things together gives

$$G^{(2)}(\tau; \tau') = e^{-\Omega(\tau - \tau') \left[ \frac{\Theta(\tau - \tau')}{1 + e^{-\beta\Omega}} - \frac{\Theta(\tau' - \tau)}{1 + e^{\beta\Omega}} \right]}.$$ 

(c) If $\beta \geq \tau > \tau' > 0$ then $G^{(2)}(\tau; \tau') = e^{-\Omega(\tau - \tau') / (1 + e^{-\beta\Omega})}$, while (since $\tau' > \tau - \beta$), $G^{(2)}(\tau - \beta; \tau') = -e^{-\Omega(\tau - \beta - \tau') / (1 + e^{\beta\Omega})} = -G^{(2)}(\tau; \tau')$. Take the limit $\tau \to \beta$ to see that, for any value of $\tau'$ in the interval $[0, \beta]$, $G^{(2)}(\beta; \tau') = -G^{(2)}(0; \tau')$. In other words, $G^{(2)}(\tau; \tau')$ may be regarded as antiperiodic in $\tau$ with period $\beta$. And since $G^{(2)}$ only depends on $\tau - \tau'$, it may equally be regarded as antiperiodic in $\tau'$ with the same period.

(d) Since $(\partial_\tau + \Omega^2)\hat{\psi}(-i\tau) = 0$, the only reason $(\partial_\tau + \Omega)G^{(2n)}(1, 2, \ldots, \tau'; 1', 2', \ldots, n')$ does not vanish identically is because of the time-ordering symbol, which causes the correlator to equal a sum of terms each of which includes a product of step functions enforcing a particular time-ordering [as seen explicitly in part (b)], and the time derivative can act on these step functions producing a temporal delta function. As a warm-up, consider the case of $G^{(2)}$:

$$(\partial_\tau + \Omega)G^{(2)}(\tau; \tau') = (\partial_\tau + \Omega) \left[ \Theta(\tau - \tau')\langle \hat{\psi}(-i\tau)\hat{\psi}^\dagger(-i\tau') \rangle - \Theta(\tau' - \tau)\langle \hat{\psi}^\dagger(-i\tau')\hat{\psi}(-i\tau) \rangle \right]$$

$$= \delta(\tau - \tau') \left[ \langle \hat{\psi}(-i\tau)\hat{\psi}^\dagger(-i\tau') \rangle + \langle \hat{\psi}^\dagger(-i\tau')\hat{\psi}(-i\tau) \rangle \right]$$

$$= \delta(\tau - \tau') \left\{ \langle \hat{\psi}(-i\tau)\hat{\psi}^\dagger(-i\tau) \rangle \right\}$$

$$= \delta(\tau - \tau'),$$

where the last step used the equal time anticommutator $\{\hat{\psi}, \hat{\psi}^\dagger\} = 1$.

For the general case, one gets a temporal delta function when the time $\tau_1$ equals the time of any other operator, multiplied by an expectation value of the equal time anticommutator of $\hat{\psi}(-i\tau_1)$ with the other operator at the same time, times a time-ordered product of the remaining $2n-2$ operators. But since $\hat{\psi}$ anticommutates with itself, only the terms involving an anticommutator of $\hat{\psi}(-i\tau_1)$ with one of the conjugate $\hat{\psi}^\dagger(-i\tau_k')$’s make non-zero contributions. Since time-ordering for fermions gives a minus sign for each interchange, the term involving $\delta(\tau_1 - \tau_k')$ gets an overall minus sign when $k$ is even, since an odd number of interchanges are required to move the $\hat{\psi}^\dagger(-i\tau_k')$ from its original position to a position just to the right of $\hat{\psi}(-i\tau_1)$. Putting these terms together gives the claimed result. Exactly the same argument may be applied to $(-\partial_{\tau'} + \Omega)G^{(2n)}$, yielding

$$(-\partial_{\tau'} + \Omega)G^{(2n)}(1 \cdots n; 1' \cdots n') = -\sum_{k=1}^{n} (-1)^{k-1} \delta(\tau_k - \tau_1') G^{(2n-2)}(1 \cdots k \cdots, n; 2' \cdots n').$$

(e) For the generalization to $N$ coupled fermionic oscillators, $\Omega$ becomes an $N \times N$ Hermitian matrix. Denote its eigenvalues by $\lambda_a$, for $a = 1, \ldots, N$. Let $|0\rangle$ denote the state annihilated by all $\hat{\psi}_a$. (If $\Omega$ is positive definite, then this is the ground state.) The spectrum of the theory consists of $2^N$ energy levels which may be labeled by an $N$-component vector $v$, each of whose
components is either 0 or 1. The eigenstate \( |v\rangle = \prod_{a=1}^{N} (\hat{\psi}_a^\dagger)^{v_a} |0\rangle \) and the corresponding energy is \( E[v] = \sum_{a=1}^{N} v_a \lambda_a \). The partition function \( Z = \text{Tr} \, e^{-\beta H} = \prod_{a=1}^{N} (1 + e^{-\beta \lambda_a}) \).

Time-ordered correlators now have extra indices,

\[
G^{(2n)}(1, 2, \ldots, n; 1', 2', \ldots, n') \equiv \left\langle \mathcal{T} \left( \psi_{a_1}(-i\tau_1) \cdots \psi_{a_n}(-i\tau_n) \bar{\psi}_{b_1}(-i\tau'_1) \cdots \bar{\psi}_{b_n}(-i\tau'_n) \right) \right\rangle.
\]

Regarding the propagator \( G^{(2)} \) as a matrix, \( G^{(2)}(\tau; \tau') \equiv \|G^{(2)}(\tau; \tau')_{ab}\| \), it has exactly the same form as the \( N = 1 \) result: \( G^{(2)}(\tau; \tau') = e^{-\Omega(\tau-\tau')} = \left[ \frac{\Theta(\tau'-\tau)}{1+e^{-\beta \tau}} \right] \left[ \frac{\Theta(\tau'-\tau)}{1+e^{\beta \tau}} \right] \), where these are now matrix exponentials and the denominators are matrix inverses. (The easiest way to prove this is to note that it is correct when \( \Omega \) is diagonal, in which case one has \( N \) decoupled oscillators. But any non-diagonal \( \Omega \) may be reduced to this decoupled case by a suitable change of variables.) The demonstration of anti-periodicity in (imaginary) time with period \( \beta \) goes through with no changes at all, and the only change in the recursion relations relating \( G^{(2n)} \) and \( G^{(2n-2)} \) is that the correlators have additional indices and each temporal delta function is now accompanied by a corresponding Kroneker delta function, from the equal time anticommutation relation \( \{ \bar{\psi}_a, \bar{\psi}_b \} = \delta_{ab} \). For example, \( \{ (\partial_\tau + \Omega) G^{(2)}(1; 1') \}_{ab} = (\partial_\tau \delta_{aa'} + \Omega_{aa'}) G^{(2)}(\tau; \tau')_{a'b} = \delta_{ab} \delta(\tau-\tau') \), etc.

4. Dirac Fermions.

Let \( S(x)_{\alpha\beta} \equiv \langle 0| \mathcal{T} (\bar{\psi}_a(x) \bar{\psi}_\beta(0)) |0\rangle = \theta(x^0) \langle 0| \bar{\psi}_a(x) \bar{\psi}_\beta(0) |0\rangle - \theta(-x^0) \langle 0| \bar{\psi}_\beta(0) \psi_a(x) |0\rangle \). Apply to \( S(x) \) the operator \( i(\gamma^\mu \partial_\mu + m) \). Since \( i(\gamma^\mu \partial_\mu + m) \psi(x) = 0 \), the only reason the result is non-zero is due to the time-ordering, which introduces step functions in time. Differentiating these produces delta functions in time, so that

\[
i(\gamma^\mu \partial_\mu + m)_{\alpha\beta} S(x)_{\beta\gamma} = (i\gamma^\gamma)_{\alpha\beta} \left[ \delta(x^0) \langle 0| \bar{\psi}_\beta(x) \bar{\psi}_\gamma(0) |0\rangle + \delta(x^0) \langle 0| \bar{\psi}_\gamma(0) \bar{\psi}_\beta(x) |0\rangle \right] = (i\gamma^\gamma)_{\alpha\beta} \langle 0| \bar{\psi}_\gamma(x), \bar{\psi}_\gamma(0) |0\rangle \delta(x^0)
\]

\[
= (i\gamma^\gamma)_{\alpha\beta} \delta^4(x) \delta(x^0)
\]

(Because of the temporal delta function, the anti-commutator of \( \psi(x) \) and \( \bar{\psi}(0) \) is an equal-time anti-commutator.) Hence, the propagator \( S(x) \) is a Green’s function for \( i(\gamma^\mu \partial_\mu + m) \).

Inserting a four-dimensional Fourier representation \( S(x) = \int d^4p/(2\pi)^4 e^{ipx} \hat{S}(p) \) into the Green’s function equation shows that \( i(ip + m) \hat{S}(p) = 1 \) or \( \hat{S}(p) = -i(ip + m)^{-1} \). This is a matrix inverse, but it is easy to do if one notices that \( (ip + m)(-ip + m) = (\hat{p})^2 + m^2 = p^2 + m^2 \). Therefore

\[
i \hat{S}(p) = (-ip + m)/(p^2 + m^2).
\]

There is just one problem: the denominator has poles on the real \( p^0 \) axis at \( p^0 = \pm \sqrt{p^2 + m^2} \). These poles are on the integration contour for the temporal Fourier transform, and hence the Fourier integral \( S(x) = \int d^4p/(2\pi)^4 e^{ipx} \hat{S}(p) \) is ill-defined. This always happens for Green’s functions of wave operators. Specifying a prescription for integrating around the poles in the Fourier representation is equivalent to selecting the Green’s function which satisfies appropriate boundary conditions at infinity. In this case, the appropriate prescription is to replace \( m^2 \rightarrow m^2 - i\epsilon \) in the denominator (just as for a free scalar field). One may justify this by inserting mode expansions and explicitly computing \( S(x) \) in the free Dirac theory (and then Fourier transforming the result). Alternatively, one can use the fact that in any quantum theory (with a ground state), time-ordered correlation functions must have analytic continuations to Euclidian space, which implies that \( \hat{S}(p) \) must be analytic in the first and third quadrants of the complex \( p^0 \) plane.

5. Majorana Fermions. If \( \psi'(x) = S \psi(x) \) for some unitary matrix \( S \), then \( (\psi'(x)^\dagger)^T = S^* (\psi^\dagger(x))^T = S^* C \psi(x) = S^* C S^{-1} \psi'(x) \). So the problem reduces to finding a unitary matrix \( S \) such that
\( S^* C S^{-1} = 1 \) or \( C = S^T S \). In our standard representation, \( C = \begin{bmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{bmatrix} \). It has two eigenvalues of +1, with corresponding eigenvectors \((0, 1, 1, 0)^T\) and \((1, 0, 0, -1)^T\) (up to arbitrary phases), and two eigenvalues of −1 with eigenvectors \((1, 0, 0, 1)^T\) and \((0, 1, -1, 0)^T\) (up to phases). Normalize these eigenvectors and stack them together to form a unitary matrix \( U \) which diagonalizes \( C \), so that \( C = U \lambda U^\dagger \) with \( \lambda \) diagonal. The matrix \( S = U^\dagger \) will be a solution to \( C = S^T S \) if we can select phases so that \( U \lambda = (U^\dagger)^T U^* \). To satisfy this, choose the eigenvectors with eigenvalue −1 to be pure imaginary, and while the eigenvectors with eigenvalue +1 are real. One explicit solution (which can be written compactly) is \( S = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & \sigma_1 \\ \sigma_1 & -i \sigma_2 \end{bmatrix} \). This change of basis induces a similarity transformation on the \( \gamma \)-matrices, \( \tilde{\gamma}_\mu \equiv S \gamma_\mu S^\dagger \) with the transformed matrices \( \tilde{\gamma}_\mu \) all being real:

\[
\begin{align*}
\tilde{\gamma}_0 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \tilde{\gamma}_1 &= \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}, & \tilde{\gamma}_2 &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, & \tilde{\gamma}_3 &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.
\end{align*}
\]