1. Bosonic Coherent State Functional Integrals.

(a) One may define the operator exponential by a power series: $\exp(\hat{a}^\dagger z) \equiv 1 + \sum_{n=1}^{\infty} z^n (\hat{a}^\dagger)^n/n!$. Using the canonical commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$ it is straightforward to prove the operator identity $\hat{a}(\hat{a}^\dagger)^n = p(\hat{a}^\dagger)^{n-1} + (\hat{a}^\dagger)^p \hat{a} (\hat{a}^\dagger)^{n-p}$ for any $p \in \{0, \cdots, n\}$. The choice $p = n$ is particularly useful when acting on the vacuum, since $\hat{a}(\hat{a}^\dagger)^n|0\rangle = n(\hat{a}^\dagger)^{n-1}|0\rangle$. It is now just a simple matter of applying this definition and identity to show that $\hat{a}|z\rangle = z|z\rangle$. This implies that bosonic coherent states are eigenstates of the annihilation operator.

(b) To calculate the overlap $\langle z'|z\rangle$ we need to understand how to move $\exp(\hat{a}\bar{z}')$ past $\exp(\hat{a}^\dagger z)$, or more fundamentally, what is the matrix element $\langle 0|\hat{a}^m (\hat{a}^\dagger)^n|0\rangle$? Repeatedly applying the identity from part (a) we find $\langle 0|\hat{a}^m (\hat{a}^\dagger)^n|0\rangle = m!\,\delta_{mn}$. Hence,

$$\begin{aligned}
\langle z'|z\rangle\, e^{(|z|^2+|z'|^2)/2} &= \langle 0|e^{\hat{a}\bar{z}'} e^{\hat{a}^\dagger z}|0\rangle \\
&= 1 + \sum_{m,n\geq 1} \frac{\bar{z}'^m z^n}{m!n!} \langle 0|\hat{a}^m (\hat{a}^\dagger)^n|0\rangle \\
&= e^{\bar{z}' z}.
\end{aligned}$$

Thus, $\langle z'|z\rangle = e^{-|z|^2/2 - |z'|^2/2 + \bar{z}' z} = e^{-[\bar{z}'(z'-z) - (\bar{z}'-\bar{z})z]/2}$. Note that $\langle z|z\rangle = 1$ so the coherent states are unit normalized. Also note that $\langle z'|z\rangle$ does not vanish when $z' \neq z$ (in contrast with harmonic oscillator eigenstates).

(c) Eigenstates of the Hamiltonian $\hat{a}^\dagger \hat{a}$ are given by $|n\rangle \equiv \frac{1}{\sqrt{n}} (\hat{a}^\dagger)^n|0\rangle$ for any nonnegative integer n. The state $|n\rangle$ has eigenvalue n and as a set they form a complete orthonormal basis for the single particle Hilbert space. The coherent state $|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle$. Therefore,

$$\int d\bar{z}dz\, |z\rangle\langle z| = \frac{1}{\pi} \sum_{m,n\geq 0} \frac{|m\rangle\langle n|}{\sqrt{m!n!}} \int d(\mathrm{Re}\, z) d(\mathrm{Im}\, z)\, e^{-|z|^2} z^m \bar{z}^n.$$

The integration measure $d(\mathrm{Re}\, z)d(\mathrm{Im}\, z) = dr\, r\, d\theta$ (with $z \equiv re^{i\theta}$), and the Gaussian $e^{-|z|^2}$, are both invariant under a phase rotation $z \to e^{i\alpha} z$. If $m \neq n$, then the product $z^m \bar{z}^n$ acquires a non-zero phase $e^{i(m-n)\alpha}$, and this implies that the integral is identically zero. The integral is nonzero only when $m = n$. Hence

$$\begin{aligned}
\int d\bar{z}dz\, |z\rangle\langle z| &= \frac{1}{\pi} \sum_{n\geq 0} \frac{|n\rangle\langle n|}{n!} (2\pi) \int_0^\infty r dr\, e^{-r^2} r^{2n} \\
&= \sum_{n\geq 0} \frac{|n\rangle\langle n|}{n!} \int_0^\infty ds\, e^{-s} s^n = \sum_{n\geq 0} |n\rangle\langle n| = 1\,,
\end{aligned}$$

where we recognized the integral representation of $\Gamma(n+1) = n!$.

This proves that any state in the Hilbert space can be written as a linear combination of coherent states, since any state $|\Psi\rangle = \int d\bar{z}dz\, |z\rangle\langle z|\Psi\rangle$. This is the usual statement of completeness. In fact, the coherent state basis is *overcomplete* because any given coherent state can be represented as a linear combination of other coherent states.

(d) The proof is very similar to part (c). Write out the matrix element in terms of eigenstates of $\hat{a}^\dagger \hat{a}$,

$$\int d\bar{z}dz\, \langle z|\mathcal{O}|z\rangle = \frac{1}{\pi} \sum_{m,n\geq 0} \frac{\langle m|\mathcal{O}|n\rangle}{\sqrt{m!n!}} \int d(\mathrm{Re}\, z) d(\mathrm{Im}\, z)\, e^{-|z|^2} z^m \bar{z}^n\,.$$

The integral is nonzero only when $m = n$ and evaluates to $\pi\, n!$. Thus,

$$\int d\bar{z}dz\, \langle z|\mathcal{O}|z\rangle = \sum_{n\geq 0} \langle n|\mathcal{O}|n\rangle = \mathrm{Tr}\,\mathcal{O}$$

(since the middle expression is just the definition of the trace using a basis of harmonic oscillator eigenstates).

(e) The partition function is

$$Z = \mathrm{Tr}\, e^{-\beta H} = \lim_{N\to\infty} \int \prod_{i=1}^{N} d\bar{z}_i dz_i\, \langle z_i|e^{-\frac{\beta}{N}\hat{H}}|z_{i-1}\rangle$$

where $z_0 \equiv z_N$. Let $\epsilon \equiv \beta/N$. Since N is being taken to infinity, ϵ is infinitesimal. Therefore,

$$\langle z'|e^{-\epsilon h(\hat{a}^\dagger,\hat{a})}|z\rangle = e^{-\epsilon h(\bar{z}',z)+O(\epsilon^2)}\langle z'|z\rangle.$$

To derive this we can expand the exponential as $e^{-\epsilon h(\hat{a}^\dagger,\hat{a})} = 1 - \epsilon h(\hat{a}^\dagger,\hat{a}) + \frac{1}{2}\epsilon^2 h(\hat{a}^\dagger,\hat{a})^2 + \cdots$. The function h is normal ordered, which means that for a given monomial term in h, all of the creation operators are to the left of the annihilation operators. One can express h explicitly as $h(\hat{a}^\dagger,\hat{a}) = \sum_{m,n} c_{m,n}(\hat{a}^\dagger)^m \hat{a}^n$.[1] Taking the Hermitian conjugate of $\hat{a}|z\rangle = z|z\rangle$ yields $\langle z|\hat{a}^\dagger = \langle z|\bar{z}$. Upon sandwiching any normal ordered monomial between coherent state bras and kets, we end up replacing $\hat{a}^\dagger$ by its left eigenvalue and $\hat{a}$ by its right eigenvalue. It is clear that $\langle z'|h|z\rangle = h(\bar{z}',z)$. However, h^2 and higher powers of h are, in general, not normal ordered. Nevertheless one can always write $h^2 = N(h^2) + (\text{stuff})$, where N is the normal ordering symbol. It follows that $\langle z'|N(h^2)|z\rangle = h(\bar{z}',z)^2$ and the full matrix element $\langle z'|h^2|z\rangle$ equals this result up to corrections coming from (stuff). This shows that it is possible to resum the expansion back into the form of an exponential of $h(\bar{z}',z)$ while making an error of order $\epsilon^2(\text{stuff})$.

Using the result of part (b) for the overlap, we have

$$Z = \lim_{N\to\infty} \int \prod_{i=1}^{N} d\bar{z}_i dz_i\, \exp\left\{\epsilon \sum_{j=1}^{N}\left[-\bar{z}_j\, \frac{z_j - z_{j-1}}{2\epsilon} + \frac{\bar{z}_j - \bar{z}_{j-1}}{2\epsilon}\, z_{j-1} - h(\bar{z}_j, z_{j-1}) + O(\epsilon)\right]\right\}.$$

Let $z(j\epsilon) = z_j$ and similarly for $\bar{z}$, and notice that

$$\lim_{\epsilon\to 0}\left[-\bar{z}_j\, \frac{z_j - z_{j-1}}{\epsilon} + \frac{\bar{z}_j - \bar{z}_{j-1}}{\epsilon}\, z_{j-1}\right] = -\bar{z}(\tau)\,[\partial_\tau z(\tau)] + [\partial_\tau \bar{z}(\tau)]\, z(\tau) \equiv -\bar{z}(\tau)\overleftrightarrow{\partial}_\tau z(\tau)\,.$$

Moreover,

$$\begin{aligned} h(\bar{z}_j, z_{j-1}) &= h(\bar{z}_j, z_j) + \epsilon\, \frac{z_{j-1} - z_j}{\epsilon}\, \partial_{z_j} h(\bar{z}_j, z_j) + \cdots \\ &\to h(\bar{z}(\tau), z(\tau)) - \epsilon\,[\partial_\tau z(\tau)]\, \partial_z h(\bar{z}(\tau), z(\tau)) + O(\epsilon^2) = h(\bar{z}(\tau), z(\tau)) + O(\epsilon)\,. \end{aligned}$$

Finally, let $\mathcal{D}\bar{z}\mathcal{D}z \equiv \lim_{N\to\infty}\prod_{i=1}^{N} d\bar{z}_i dz_i$ and recognize that $\epsilon\sum_{j=1}^{N} \to \int_0^\beta d\tau$. Any terms proportional to ϵ in the exponent vanish in the limit. Since $z_N \equiv z_0$, the complex path $z(\tau)$ obeys $z(\beta) = z(0)$, and similarly for $\bar{z}$. In other words, they are periodic with period β. Thus,

$$Z = \int_{\text{periodic paths}} \mathcal{D}\bar{z}\,\mathcal{D}z\, e^{-\int_0^\beta d\tau\,\left[\frac{1}{2}\bar{z}(\tau)\overleftrightarrow{\partial}_\tau z(\tau) + h(\bar{z}(\tau),z(\tau))\right]}.$$

[1] For example, the simple harmonic oscillator with a $\lambda\hat{x}^4$ anharmonic perturbation has a normal ordered Hamiltonian of the form $h = \hat{a}^\dagger\hat{a} + \lambda[(\hat{a}^\dagger)^4 + 4(\hat{a}^\dagger)^3\hat{a} + 6(\hat{a}^\dagger)^2\hat{a}^2 + 4\hat{a}^\dagger\hat{a}^3 + \hat{a}^4 + 6(\hat{a}^\dagger)^2 + 12\hat{a}^\dagger\hat{a} + 6\hat{a}^2 + 3]$.

(f) Separate $z(\tau) = \mathrm{Re}\, z(\tau) + i\mathrm{Im}\, z(\tau)$. Instead of treating z and $\bar{z}$ as the integration variables, treat $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ as the independent degrees of freedom. A simple rewriting gives

$$Z = \int\limits_{\text{periodic paths}} \mathcal{D}(\mathrm{Re}\, z)\mathcal{D}(\mathrm{Im}\, z)\, e^{-\int_0^\beta d\tau\, [i(\mathrm{Re}\, z)\partial_\tau \mathrm{Im}\, z - i(\partial_\tau \mathrm{Re}\, z)\mathrm{Im}\, z + h(\mathrm{Re}\, z, \mathrm{Im}\, z)]}.$$

Let $z = (\varphi + i\pi)/\sqrt{2}$, with φ and π real. For a harmonic oscillator with unit frequency, $\hat{a} = (\hat{x} + i\hat{p})/\sqrt{2}$, so φ would be the coherent state expectation value of the position, and π the coherent state expectation value of momentum. In the exponent of the path integral, integrate the first term by parts to flip the time derivative onto the other factor (and use the fact that both real and imaginary parts of z are periodic to see that the boundary term vanishes), to obtain the natural form of a phase space functional integral (with imaginary time),

$$Z = \int\limits_{\text{periodic paths}} \mathcal{D}\varphi\, \mathcal{D}\pi\, e^{-\int_0^\beta d\tau\, [i\pi(\tau)\partial_\tau \varphi(\tau) + h(\pi, \varphi)]}.$$

Suppose that the Hamiltonian has the typical form of a sum of kinetic and potential energy, so that $h(\pi, \varphi) = \frac{1}{2}\pi^2 + V(\varphi)$. Then the functional integral over $\pi(\tau)$ is just a (shifted) Gaussian integral. Performing this integral reduces the phase space functional integral to a conventional (Euclidean) coordinate space path integral,

$$Z = \int\limits_{\text{periodic paths}} \mathcal{D}\varphi\, \mathcal{D}\pi\, e^{-\int_0^\beta d\tau\, \left[i\pi(\tau)\partial_\tau\varphi(\tau) + \frac{1}{2}\pi(\tau)^2 + V(\varphi(\tau))\right]} = \int\limits_{\text{periodic paths}} \mathcal{D}\varphi\, e^{-\int_0^\beta d\tau\, \left[\frac{1}{2}(\partial_\tau\varphi(\tau))^2 + V(\varphi(\tau))\right]}.$$

2. Fermionic Coherent State Functional Integrals.

(a) In this problem, $\{\bar{z}, z\}$ are generators of a Grassmann algebra. When we act with $\hat{b}$ on $|z\rangle$ we may move it past $e^{-\bar{z}z/2}$ since the latter is an even element of the Grassmann algebra. Since $e^{\hat{b}^\dagger z} = 1 + \hat{b}^\dagger z$, the only nonvanishing term is $e^{-\bar{z}z/2}\{\hat{b}, \hat{b}^\dagger\}z|0\rangle$. The anticommutator is just 1 and $z|0\rangle = z(1 + \hat{b}^\dagger z)|0\rangle = (1 + \hat{b}^\dagger z)|0\rangle z$.[2] So the final result is $\hat{b}|z\rangle = z|z\rangle = |z\rangle z$.

(b) Define the Hermitian conjugate so that $(\hat{b}^\dagger z')^\dagger = \bar{z}'\hat{b}$. Then $\langle z'| = \langle 0|(1 + \bar{z}'\hat{b})e^{-\bar{z}'z'/2}$.

$$\begin{aligned}
\langle z'|z\rangle &= e^{-\bar{z}'z'/2}\langle 0|(1 + \bar{z}'\hat{b})|z\rangle \\
&= e^{-\bar{z}'z'/2}\underbrace{(1 + \bar{z}'z)}_{e^{\bar{z}'z}}\underbrace{\langle 0|z\rangle}_{e^{-\bar{z}z/2}} \\
&= e^{-\bar{z}z/2 - \bar{z}'z'/2 + \bar{z}'z} = e^{-\frac{1}{2}\bar{z}'(z'-z) + \frac{1}{2}(\bar{z}'-\bar{z})z}.
\end{aligned}$$

The answer is completely analogous to the bosonic case. Notice that $\langle z|z\rangle = 1$, but $\langle z'|z\rangle \neq 0$ when $z' \neq z$.

(c) The standard rule for Grassmann integration can be summed up as $\int d\bar{z}dz(\alpha + \beta z + \gamma\bar{z} + \delta z\bar{z}) = \delta$, where $\alpha, \beta, \gamma, \delta$ are ordinary numbers. For the quantum mechanics of a single fermionic

[2] Both equalities hold since two z's kill each other regardless of whether $\hat{b}^\dagger$ commutes or anticommutes with z.

degree of freedom, the Hilbert space consists of just two states: $\{|0\rangle, |1\rangle \equiv \hat{b}^\dagger|0\rangle\}$. Therefore,

$$\begin{aligned}
\int d\bar{z}dz\,|z\rangle\langle z| &= \int d\bar{z}dz\,e^{-\bar{z}z}\,(|0\rangle + |1\rangle z)(\langle 0| + \bar{z}\langle 1|)\\
&= \int d\bar{z}dz\,(1+z\bar{z})\,(|0\rangle\langle 0| + |0\rangle\bar{z}\langle 1| + |1\rangle z\langle 0| + |1\rangle z\bar{z}\langle 1|)\\
&= \int d\bar{z}dz\,z\bar{z}\,(|0\rangle\langle 0| + |1\rangle\langle 1|)\\
&= |0\rangle\langle 0| + |1\rangle\langle 1|\\
&= 1.
\end{aligned}$$

(d) The proof is straightforward,

$$\begin{aligned}
\int d\bar{z}dz\,\langle -z|\mathcal{O}|z\rangle &= \int d\bar{z}dz\,e^{-\bar{z}z}\,(\langle 0| - \bar{z}\langle 1|)\,\mathcal{O}\,(|0\rangle + |1\rangle z)\\
&= \int d\bar{z}dz\,(1+z\bar{z})\,(\langle 0|\mathcal{O}|0\rangle + \langle 0|\mathcal{O}|1\rangle z - \bar{z}\langle 1|\mathcal{O}|0\rangle - \bar{z}\langle 1|\mathcal{O}|1\rangle z)\\
&= \int d\bar{z}dz\,z\bar{z}\,(\langle 0|\mathcal{O}|0\rangle + \langle 1|\mathcal{O}|1\rangle)\\
&= \mathcal{O}_{00} + \mathcal{O}_{11}\\
&= \mathrm{Tr}\,\mathcal{O}.
\end{aligned}$$

Note that $\int d\bar{z}dz\,\langle z|\mathcal{O}|z\rangle = \mathcal{O}_{00} - \mathcal{O}_{11} = \mathrm{Tr}(-1)^F\mathcal{O}$. Here $(-1)^F|0\rangle \equiv |0\rangle$ but $(-1)^F|1\rangle \equiv -|1\rangle$.

(e) The partition function is

$$Z = \mathrm{Tr}\,e^{-\beta\hat{H}} = \lim_{N\to\infty}\int\prod_{i=1}^{N} d\bar{z}_i dz_i\,\langle z_i|e^{-\frac{\beta}{N}\hat{H}}|z_{i-1}\rangle$$

where $z_N \equiv -z_0$. Let $\epsilon \equiv \beta/N$. The most general Hamiltonian for a single fermionic degree of freedom is $\hat{H} = h(\hat{b}^\dagger, \hat{b}) = c_0 + c_1(\hat{b} + \hat{b}^\dagger) + c_2\hat{b}^\dagger\hat{b}$. Since the expansion of the exponential truncates, the matrix element is easily checked to be

$$\langle z'|e^{-\epsilon h(\hat{b}^\dagger,\hat{b})}|z\rangle = e^{-\epsilon h(\bar{z}',z)+O(\epsilon^2)}\langle z'|z\rangle.$$

Inserting the overlap from part (b) we get an expression for Z that is superficially identical to the one for the bosonic case. We can follow all of the steps in part (e) of the previous problem without any cosmetic changes, but keeping in mind that, behind the scenes, the integration variables are Grassmann generator-valued functions of time. The other crucial difference is that $z(\beta) = -z(0)$. Thus, the paths given by $z(\tau)$ and $\bar{z}(\tau)$ are antiperiodic, rather than periodic, in Euclidean time. Hence,

$$Z = \int_{\text{antiperiodic paths}} \mathcal{D}\bar{z}\mathcal{D}z\,e^{-\int_0^\beta d\tau\,\left[\frac{1}{2}\bar{z}(\tau)\overleftrightarrow{\partial}_\tau z(\tau) + h(\bar{z}(\tau),z(\tau))\right]}.$$

3. (a) $Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\,e^{-S_E[\psi]}$, with $S_E[\psi] \equiv \int_0^\beta dx^0 \int_0^L d^3x\,\bar{\psi}(\gamma_E^\mu\partial_\mu + m)\psi$, where the integral is over all Grassmann-valued spinors $\psi(x)$ and $\bar{\psi}(x)$ which are antiperiodic in the (imaginary) time coordinate x^0 with period β and periodic in x^i with period L, so $\psi(0,\underset{\sim}{x}) = -\psi(\beta,\underset{\sim}{x})$ and $\psi(x^0,\underset{\sim}{x}+\underset{\sim}{n}L) = \psi(x^0,\underset{\sim}{x})$, and likewise for $\bar{\psi}(x)$, for any integer-valued vector $\underset{\sim}{n}$.

(b) The functional integral is a complex Grassmann Gaussian integral, $Z = \int \mathcal{D}\bar\psi \mathcal{D}\psi \, e^{-(\bar\psi, \mathcal{O}\psi)}$, with inner product $(f,g) = \int_0^\beta dx^0 \int_0^L d^3x \, f(x) g(x)$ and covariance $\mathcal{O} = \gamma_E^\mu \partial_\mu + m$. Therefore $Z = \det(\mathcal{O})$ and the associated free energy $F \equiv -\beta^{-1} \ln Z = -\beta^{-1} \ln \det(\mathcal{O})$, where the determinant is to be evaluated in the space of functions which are antiperiodic in time with period β and periodic in space with period L.

(c) Eigenfunctions of $\mathcal{O} = \not\partial + m$ satisfying the periodicity conditions are four-dimensional plane waves $e^{ik^0 x^0 + i\underset{\sim}{k}\cdot\underset{\sim}{x}}$ times spinors u_k which are eigenvectors of $i\not k + m$, where k^0 is an odd integer multiple of π/β and each k^i is an integer multiple of $2\pi/L$. Since $(i\not k)^2 = -k^2$, the eigenvalues of $i\not k + m$ are $\pm i|k| + m$, where $|k| = \sqrt{k^2}$ is the Euclidean norm of k. Because $\not k$ is traceless, there must be two degenerate eigenvalues of $i|k| + m$, and two of $-i|k| + m$.

So the free energy $F = -\beta^{-1} \ln \det(\not\partial + m) = -\beta^{-1} \sum_k [2 \ln(i|k| + m) + 2 \ln(-i|k| + m)] = -2\beta^{-1} \sum_k \ln(k^2 + m^2)$, where the sum runs over the allowed values of the Euclidean 4-vector k which satisfy the periodicity conditions. Writing k^0 explicitly as $\omega_n \equiv (2n+1)\pi/\beta$, for n integer, and the allowed values of $\underset{\sim}{k}$ as $\underset{\sim}{k}_{\underset{\sim}{l}} \equiv (2\pi/L)\, \underset{\sim}{l}$, for $\underset{\sim}{l}$ an integer-valued vector, we have $F = -2\beta^{-1} \sum_{\underset{\sim}{l}} \sum_n \ln(\omega_n^2 + \underset{\sim}{k}_{\underset{\sim}{l}}^2 + m^2)$. The sum doesn't converge, but taking a derivative with respect to m^2 improves things since $\frac{\partial}{\partial m^2} F = -2\beta^{-1} \sum_{\underset{\sim}{l}} \sum_n (\omega_n^2 + \underset{\sim}{k}_{\underset{\sim}{l}}^2 + m^2)^{-1}$, and now the first sum over n does converge.

(d) Using $\sum_{n=-\infty}^{\infty} \left[(2n+1)^2 + b^2\right]^{-1} = \frac{\pi}{2b} \tanh \frac{\pi b}{2}$, we have $\frac{\partial}{\partial m^2} F = -\sum_{\underset{\sim}{l}} \epsilon(\underset{\sim}{k}_{\underset{\sim}{l}})^{-1} \tanh(\frac{1}{2}\beta\epsilon(\underset{\sim}{k}_{\underset{\sim}{l}}))$, where $\epsilon(\underset{\sim}{k}) \equiv \sqrt{\underset{\sim}{k}^2 + m^2}$. Now $\tanh(\frac{1}{2}\beta\epsilon) = \frac{e^{\frac{1}{2}\beta\epsilon} - e^{-\frac{1}{2}\beta\epsilon}}{e^{\frac{1}{2}\beta\epsilon} + e^{-\frac{1}{2}\beta\epsilon}} = 1 - 2n_f(\epsilon)$ with $n_f(\epsilon) \equiv 1/(e^{\beta\epsilon} + 1)$ the usual equilibrium Fermi distribution function. Therefore $\frac{\partial}{\partial m^2} F = \sum_{\underset{\sim}{l}} g(\underset{\sim}{k}_{\underset{\sim}{l}})$ with $g(\underset{\sim}{k}) = \frac{1}{\epsilon(\underset{\sim}{k})} [2n_f(\epsilon(\underset{\sim}{k})) - 1]$. This is a smooth function of its argument. If this sum is multiplied by $(2\pi/L)^3$ (which is the cube of the spacing between allowed momenta) the result is a Riemann sum approximation to the three dimensional momentum integral $\int d^3k \; g(\underset{\sim}{k}) = \lim_{L\to\infty} (\frac{2\pi}{L})^3 \sum_{\underset{\sim}{l}} g(\underset{\sim}{k}(\underset{\sim}{l}))$, or equivalently $\sum_{\underset{\sim}{l}} g(\underset{\sim}{k}(\underset{\sim}{l})) \to L^3 \int \frac{d^3k}{(2\pi)^3} \, g(\underset{\sim}{k})$ for large L. Therefore, in the large volume limit, $\frac{\partial}{\partial m^2} F = L^3 \int \frac{d^3k}{(2\pi)^3} \frac{1}{\epsilon(\underset{\sim}{k})} [2n_f(\epsilon(\underset{\sim}{k})) - 1]$. Note that this differs from the previous result for a real scalar field by a replacement of the Bose distribution by the Fermi distribution, an overall factor of four accounting for the two spin states of our spin-$\frac{1}{2}$ fermions plus two for the antifermions, and a change in sign of the vacuum energy contribution.

(e) Integrating $\frac{\partial}{\partial m^2} F$ with respect to m^2 gives the free energy density $F/L^3 = -\int \frac{d^3k}{(2\pi)^3} \, \{2\epsilon(\underset{\sim}{k}) + 4\beta^{-1} \ln[1 + e^{-\beta\epsilon(\underset{\sim}{k})}]\} + C$, for some integration constant C. The first term in the integrand, $2\epsilon(\underset{\sim}{k})$, is a temperature independent zero-point energy. To build a Lorentz invariant theory, we want to use our freedom to add a constant to the Lagrange density (and hence, a constant times βL^3 to the action) to make the vacuum energy vanish. Therefore, we should simply drop the $2\epsilon(\underset{\sim}{k})$ term in the integrand. The integration constant C must be independent of m^2, and must have dimensions of (energy)^4. The only possibility is that it is some pure number times T^4 (where the temperature $T \equiv \beta^{-1}$). Such a contribution, depending on no mass scale except the temperature, would indicate the presence of some sort of blackbody radiation. No such contribution should exist for a theory which only contains massive particles, so C should be set to zero. The result is that the physical free energy density is $F/L^3 = -4\beta^{-1} \int \frac{d^3k}{(2\pi)^3} \, \ln[1 + e^{-\beta\epsilon(\underset{\sim}{k})}]$. This is a well-defined convergent integral. The corresponding energy density $E/L^3 = \frac{\partial}{\partial\beta}(\beta F/L^3) = 4\int \frac{d^3k}{(2\pi)^3} \, \epsilon(\underset{\sim}{k}) \, [e^{\beta\epsilon(\underset{\sim}{k})} + 1]^{-1} = 4\int \frac{d^3k}{(2\pi)^3} \, \epsilon(\underset{\sim}{k}) \, n_f(\epsilon(\underset{\sim}{k}))$. This is the correct result — equal to an integral over all modes of the energy $\epsilon(\underset{\sim}{k})$ of the mode times the occupation number of the mode, given by the Fermi distribution function, with an overall factor of four for spin and antiparticles.