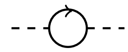
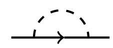
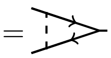
1. Yukawa theory (A).

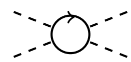
   (a) If $\phi \to -\phi$ and $\psi \to A\psi$ then, recalling that $\bar\psi \equiv \psi^\dagger(i\gamma^0)$, the conjugate fermion transforms as $\bar\psi \to \bar\psi(i\gamma^0)A^\dagger(i\gamma^0)$. For the Yukawa coupling term $\bar\psi\,\phi\,\psi$ to be invariant, this transformation of $\bar\psi$ must equal $-\bar\psi A^{-1}$, implying that $A^{-1} = -(i\gamma^0)A^\dagger(i\gamma^0)$. This will be satisfied if A anticommutes with γ^0 and is unitary (so that $A^\dagger = A^{-1}$). For the fermion kinetic term $\bar\psi \not\partial \psi$ to also be invariant, A must anticommute with all the gamma matrices. Choosing $A = \gamma_5$ does the job. (My γ_5 is Hermitian and squares to one.) This is called a *chiral* transformation because, if ψ is decomposed into right and left handed pieces, this transformation transforms the two pieces differently.

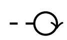
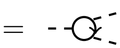
   (b) The UV divergent one-loop diagrams are:

      i. The scalar self-energy $\Sigma_\phi(k) =$  $= -g^2 \int \frac{d^4p}{(2\pi)^4}\, \mathrm{tr}\,[\not p(\not p + \not k)]\ /\ [k^2(p+k)^2]$.

      ii. The fermion self-energy $\Sigma_\psi(p) =$  $= ig^2 \int \frac{d^4k}{(2\pi)^4}\, (\not p + \not k)\ /\ [(k^2+m^2)(p+k)^2]$.

      iii. The fermion vertex correction $\Gamma =$  $\simeq g^3 \int \frac{d^4l}{(2\pi)^4}\, \not l \not l\ /\ [(l^2)(l^2)(l^2+m^2)]$.

      iv. The quartic scalar diagram $G^{(4)} =$  + perms. $\simeq -6g^4 \int \frac{d^4l}{(2\pi)^4}\, \mathrm{tr}\,[\not l \not l \not l \not l]\ /\ (l^2)^4$.

   The scalar tadpole diagram $\Gamma_1 =$  and the cubic scalar diagram $\Gamma_3 =$  would also, by power counting, appear to be divergent. But these diagrams involve a trace of an odd number of gamma-matrices, and hence vanish identically. In these diagrams, dashed lines denote the free scalar propagator while directed solid lines denote a free fermion propagator which (in momentum space) is just $-i\not p/p^2$ since there is no fermion bare mass. (The above expressions for Γ and $G^{(4)}$ omit the external momenta which add to the loop momentum on some propagators. Only the above leading high (loop) momentum forms will be needed below.)

   Regarding the overall coefficients above, note that each vertex (in Euclidean space) carries a factor of $-g$, each fermion propagator brings a $-i$, and every closed fermion loop receives an additional minus sign. Beyond that, self-energies are defined as minus the 1PI diagram, so that a positive self-energy correction gives a positive correction to the mass (or mass-squared) when resummed into the propagator denominator. The factor of 6 in $G^{(4)}$ accounts for the six permutations of external lines each of which represent distinct contributing diagrams. Because the fermion lines are directed, all symmetry factors are unity.

   The scalar self-energy has dimension two and is quadratically divergent (or UV sensitive) so with a momentum cutoff Λ it can have the form

$$\Sigma_\phi(k) = c_1\, g^2\, \Lambda^2 + c_2\, g^2\, m^2\, \ln \Lambda/\mu + c_3\, g^2\, k^2\, \ln \Lambda/\mu + \text{finite}\,,$$

   with each c_i coefficient equal to some pure number. The k-independent c_1 and c_2 terms can be absorbed by suitably adjusting the scalar bare mass m^2. The quadratic (in k) c_3 term can be absorbed by scalar field wavefunction renormalization.

   The fermion self-energy has dimension one and appears, by power counting, to be linearly divergent implying that it could have the form

$$\Sigma_\psi(p) = a_1\, g^2\, \Lambda + a_2\, g^2\, m\, \ln \Lambda/\mu + a_3\, g^2\, \not p\, \ln \Lambda/\mu + \text{finite}\,,$$

   with each a_i equal to a pure number. However, the p-independent a_1 and a_2 terms, which multiply a 4×4 identity matrix (with Dirac indices) must actually vanish. One way to understand this is to note that if these terms did not vanish, then their UV sensitivity could only be absorbed by adding, and suitably adjusting, a fermion bare mass. But adding such a fermion mass would violate the chiral symmetry of the theory. Alternatively, at a hands-on level, the integrand of the above expression for the fermion self-energy is a linear combination of gamma matrices, so the result of the integral must also be some linear combination of $\{\gamma^\mu\}$; it cannot produce any term proportional to the identity matrix. As p is only spacetime vector which the result can depend on, the answer must be $g^2\,\not{p}$ times some function of p^2. The net result is that the fermion self-energy only has only logarithmic UV sensitivity proportional to $\not{p}$, which may be absorbed by fermion wavefunction renormalization.

The fermion vertex correction Γ is dimensionless and, by power counting logarithmically divergent, so it must have the form

$$\Gamma(p,p';k) = b\,g^3\,\ln\Lambda/\mu + \text{finite}\,,$$

with b some pure number. This UV sensitivity may be absorbed by suitable adjusting the bare Yukawa coupling g. The quartic scalar diagram Γ_4 is dimensionless and, by power counting, logarithmically divergent, so it must have the form

$$G^{(4)}(\{k_i\}) = d\,g^4\,\ln\Lambda/\mu + \text{finite}\,,$$

with d some pure number. This UV sensitivity cannot be absorbed by adjusting any of the terms in the initial action, rather one *must* add an additional interaction to the theory, namely a quartic scalar $\lambda\phi^4$ interaction. Doing so is necessary so that the UV sensitivity in G_4 may be absorbed by suitably adjusting the bare quartic coupling λ. However, once this is done there will be additional UV divergent one-loop diagrams, namely the usual scalar ϕ^4 theory self-energy and four point 1PI diagrams, with the latter having the form

$$G^{(4)}(\{k_i\})\big|_{\phi^4\text{ contribution}} = d'\,\lambda^2\,\ln\Lambda/\mu + \text{finite}\,,$$

with d' some pure number. These additional UV sensitivities may be absorbed by performing the usual scalar theory renormalizations, namely suitable adjustments of the scalar bare mass m^2, quartic coupling λ, and scalar field wavefunction renormalization. The resulting theory is perturbatively renormalizable.

(c) The renormalization group (RG) equations for the dimensionless couplings g and λ arise from the above logarithmic UV sensitivities, since these automatically imply logarithmic dependence on the renormalization point μ. The vertex correction Γ adds to the tree-level vertex of $-g$ in the $\bar\psi\psi\phi$ three-point function, so the required adjustement of the UV-cutoff dependent bare Yukawa coupling is $g(\Lambda) = g_{\rm ren}(\mu) + b\,g_{\rm ren}^3\ln\Lambda/mu + \cdots$, where $\cdots$ represents higher order corrections of order g^5 or $g^3\lambda$. Demanding that

$$0 = \mu\tfrac{d}{d\mu}\,g(\Lambda) = \mu\tfrac{d}{d\mu}\left[g_{\rm ren}(\mu) + b\,g_{\rm ren}^3\ln\Lambda/\mu + \cdots\right]$$

gives $\mu\frac{d}{d\mu}\,g_{\rm ren}(\mu) = b\,g_{\rm ren}^3$, up to higher order corrections. In the scalar four-point function, the tree-level quartic interaction vertex of $-\lambda(\mu)$ receives corrections from both fermion and scalar one-loop contributions, so the required adjustment of the UV-cutoff dependent bare quartic coupling is $\lambda(\Lambda) = \lambda_{\rm ren}(\mu) + (d\,g^4 + d'\,\lambda^2)\,\ln\Lambda/\mu + \cdots$. Demanding that

$$0 = \mu\tfrac{d}{d\mu}\,\lambda(\Lambda) = \mu\tfrac{d}{d\mu}\left[\lambda_{\rm ren}(\mu) + (d\,g_{\rm ren}^4 + d'\,\lambda_{\rm ren}^2)\,\ln\Lambda/\mu + \cdots\right]$$

gives $\mu\frac{d}{d\mu}\,\lambda = d\,g^4 + d'\,\lambda^2$ up to higher order corrections.

To evaluate the various coefficients, note that the high-momentum forms of the various loop integarls above all reduce to multiples the same basic form:

$$\Gamma = \tfrac{1}{4}\,g^3\,I\,,\quad G^{(4)}_{\rm fermi} = -24\,g^4\,I\,,\quad G^{(4)}_{\rm scalar} = \tfrac{3}{2}\,\lambda^2\,I\,,$$

with $I \equiv \int_\mu^\Lambda \frac{d^4 l}{(2\pi)^4}\, l^{-4} = (8\pi^2)^{-1} \ln \Lambda/\mu$. The factor of $\frac{1}{4}$ in the vertex correction Γ comes from 4D angular averaging of $l^\mu l^\nu$, the additional factor of 4 in $G^{(4)}_{\rm fermi}$ is from the Dirac trace, and the factor of $\frac{3}{2}$ in $G^{(4)}_{\rm scalar}$ is from the three distinct 1PI scalar four-point diagrams (differing just by permutations of external lines) times a symmetry factor of $\frac{1}{2}$ for each diagram. Putting things together, we have the coupled pair of one-loop RG equations:

$$\mu \frac{d}{d\mu}\, g_{\rm ren}(\mu) = \frac{g_{\rm ren}^3}{32\pi^2}\,, \qquad \mu \frac{d}{d\mu}\, \lambda_{\rm ren}(\mu) = \frac{3}{16\pi^2}\left(\lambda_{\rm ren}^2 - 16 g_{\rm ren}^4\right).$$

(d) A continuous symmetry phase rotating a complex scalar field, $\phi \to e^{i\alpha}\,\phi$, is the same as a two-dimensional rotation: $\begin{pmatrix}{\rm Re}\phi \\ {\rm Im}\phi\end{pmatrix} \to \begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}{\rm Re}\phi \\ {\rm Im}\phi\end{pmatrix}$. To construct a Yukawa theory with a continuous chiral symmetry, consider the pair of Lorentz invariant fermion bilinears $\bar\psi\psi$ and $i\bar\psi\gamma_5\psi$, and define a transform which will cause these two bilinears to similarly rotate into each other, namely $\psi \to e^{i\beta\gamma_5}\psi$. Since γ_5 squares to one, this is the same as $\psi \to (\cos\beta + i\sin\beta\,\gamma_5)\,\psi$. Under this transformation $\begin{pmatrix}\bar\psi\psi \\ i\bar\psi\gamma_5\psi\end{pmatrix} \to \begin{pmatrix}\cos 2\beta & \sin 2\beta \\ -\sin 2\beta & \cos 2\beta\end{pmatrix}\begin{pmatrix}\bar\psi\psi \\ i\bar\psi\gamma_5\psi\end{pmatrix}$, while $\bar\psi\gamma^\mu\psi$ is unchanged. So an interaction term involving the dot product $\begin{pmatrix}\bar\psi\psi \\ i\bar\psi\gamma_5\psi\end{pmatrix}\cdot\begin{pmatrix}{\rm Re}\phi \\ {\rm Im}\phi\end{pmatrix}$ will be invariant under the continuous chiral transformation taking $\psi \to e^{i\beta\gamma_5}\,\psi$ and $\phi \to e^{2i\beta}\,\phi$. The complete (renormalizable) complex Yukawa theory Lagrange density with such a symmetry is

$$\mathcal{L} = |\partial\phi|^2 + m^2|\phi|^2 + \tfrac{1}{4}\,\lambda|\phi|^4 + \bar\psi\big(\partial\!\!\!/ + g\,({\rm Re}\phi + i\gamma_5\,{\rm Im}\phi)\big)\psi + \text{(const.)}.$$

2. Yukawa theory (B).

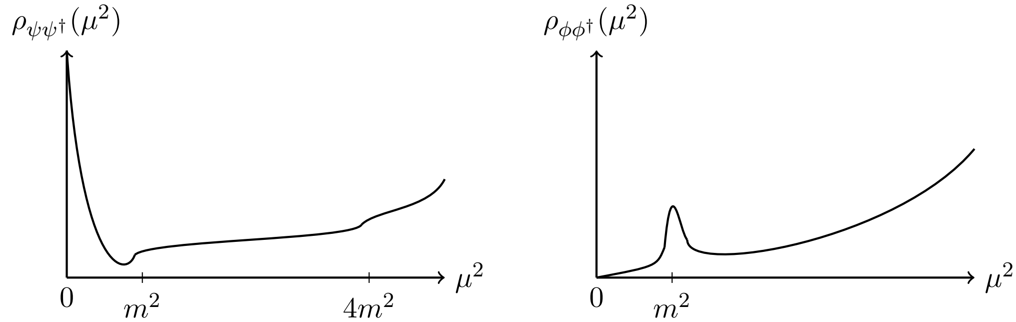
(a) The theory will contain a stable massless fermion. Because of the chiral symmetry, interactions cannot generate any mass for this fermion.[1] And likewise there is a stable antifermion. These states must be stable as fermion number is conserved and these are the lightest states with fermion numbers ± 1, so there is nothing that these states could possibly decay into. In the free theory with no Yukawa interaction there would also be a stable scalar particle of mass m. But in the presence of the Yukawa interaction, this scalar can decay into a fermion-antifermion pair. So, for weak coupling, $g \ll 1$, there will be an unstable scalar resonance with mass about m and width of order $g^2 m$.

(b) Since the fermion is massless, single fermion states will contribute to the fermion spectral density at $p^2 = 0$ (or in other words, at $p^0 = \pm|\underset{\sim}{p}|$). Because these are massless particles, odd-numbered multi-particle states can also begin contributing to the spectral density starting right at $p^2 = 0$, specifically three particle $ff\bar f$ states consisting of two fermions plus one antifermion, five particle states $fff\bar f\bar f$ states, etc. Since there is no separation between single particle and multiparticle contributions, the actual behavior of the spectral density in the neighborhood of $p^2 = 0$ is not just a single particle delta function, it is actually the discontinuity of some form of branch cut. The resulting fermion spectral density will be non-zero for all $p^2 > 0$, but should show a smoothed approximate threshold at $p^2 \approx m^2$ corresponding to an enhanced contribution of $ff\bar f$ states which are well-described as a ϕ resonance plus a separate massless fermion, with further approximate (smoothed) thresholds at $p^2 \approx 4m^2$, $9m^2$, etc., corresponding to enhanced contributions from two (or three, etc.) ϕ resonances plus a fermion.

[1]Unless this symmetry becomes spontaneously broken due to the development of a non-zero vacuum expectation value for the scalar ϕ. But the problem said to consider $m^2 > 0$ for the scalar, so this logical possibility is ignored.

The scalar spectral density will show a resonance at $p^2 \approx m^2$. Because the fermions which the resonance decays into are massless, this spectral density will be non-zero for all $p^2 > 0$ reflecting the contribution of two-particle fermion-antifermion states.

The following are schematic sketches of these spectral densities.[2]



(c) Physical processes in this theory are:

i. Two fermion elastic scattering, $ff \to ff$, via ϕ exchange,

ii. Two antifermion elastic scattering, $\bar{f}\bar{f} \to \bar{f}\bar{f}$, via ϕ exchange,

iii. Fermion-antifermion elastic scattering, involving either ϕ exchange or virtual production of a ϕ resonance,

iv. Inelastic scattering involving pair production of one or more additional fermion-antifermion pairs. For example, $ff \to fff\bar{f}$.

The only $1 \to 2$ "particle" decay process is the decay of the ϕ resonance, which is not a stable particle — but can be produced in fermion-antifermion scattering with the appropriate center-of-mass energy. No two to three particle scattering processes are possible (where "particle" means the genuinely stable fermion or antifermion), although $2 \to 4$ scattering processes like $ff \to fff\bar{f}$ will show a resonant enhancement near $E^{\text{tot}}_{\text{c.m.}} \approx m$ which may be interpreted as production of the the ϕ resonance via $ff \to ff\phi$ followed by decay of the resonance into a fermion-antifermion pair.

[2] Spectral densities may be defined (in any translation invariant theory) as a Fourier transform of the commutator, $\chi(q) = \int d^4y\, e^{-iq\cdot(y-x)}\, \langle 0|\, [\hat{A}(y),\, \hat{A}^\dagger(x)]\, |0\rangle$. But in a Lorentz invariant theory, a spectral density can only depend on Lorentz invariants, namely q^2 and $\text{sgn}(q^0)$, so that it must have the form $\chi(q) = \Theta(q^0)\, \rho_+(-q^2) - \Theta(-q^0)\, \rho_-(-q^2)$. If the operator A is either CPT even or CPT odd, one may show that this implies that $\rho_+ = \rho_-$, so that $\chi(q) = \text{sgn}(q^0)\, \rho(-q^2)$, with $i\,\tilde{G}(p) = \int_0^\infty \frac{d\mu^2}{2\pi}\, \rho(\mu^2)/(p^2 + \mu^2 - i\epsilon)$ the resulting relation between a time-ordered correlator and its Lorentz invariant spectral density.