1. (a) We shall work in Euclidean space where all the gamma matrices are Hermitian. Since γ_5 is Hermitian it may be diagonalized by some unitary matrix V. $\operatorname{tr}\gamma_5 = 0$ and $\gamma_5^2 = 1$ imply that γ_5 has eigenvalues ± 1 each with multiplicity two. Thus,

$$Ve^{i\alpha\gamma_5}V^\dagger = e^{i\alpha V\gamma_5 V^\dagger} = \begin{pmatrix} e^{i\alpha}\mathbb{1} & 0 \\ 0 & e^{-i\alpha}\mathbb{1} \end{pmatrix} = \cos\alpha + iV\gamma_5 V^\dagger \sin\alpha.$$

Multiplying by $V^\dagger$ on the left and V on the right yields $e^{i\alpha\gamma_5} = \cos\alpha + i\gamma_5\sin\alpha$. This derivation does not require a specific choice of representation for the gamma matrices. Another acceptable way to derive this equation is to write the Taylor series definition of the exponential and consider the terms with even and odd powers separately.

If $\alpha = \pi/2$, then $\psi \to i\gamma_5\psi$, $\psi^\dagger \to \psi^\dagger(-i\gamma_5)$. In a basis where γ_5 is diagonal (sometimes called the "Weyl" or "chiral" basis) the action of the chiral transformation is easily understood: the upper (two-component) spinor of ψ is multiplied by $+i$, but the lower spinor of ψ is multiplied by $-i$.

If $\alpha = \pi$, then the transformation amounts to a sign flip for the entire Dirac spinor, $\psi \to -\psi$, $\psi^\dagger \to -\psi^\dagger$.

(b) Under a continuous chiral transformation, $\bar\psi\partial\!\!\!/\psi \to \psi^\dagger e^{-i\alpha\gamma_5}\gamma^0\gamma^\mu\partial_\mu e^{i\alpha\gamma_5}\psi$. Since γ_5 commutes with the product of an even number of gamma matrices, it follows that we can commute either exponential past $\gamma^0\gamma^\mu$. The exponentials cancel when they come together and we get back $\bar\psi\partial\!\!\!/\psi$. This shows that the Dirac fermion kinetic term is invariant under chiral transformations.

(c) Note that $\gamma^0 e^{-i\alpha\gamma_5}\gamma^0 = e^{i\alpha\gamma_5}$ because γ^0 commutes past an even power of γ_5, but anticommutes past an odd power of γ_5. Therefore,

$$\begin{aligned}\bar\psi\psi &\to \psi^\dagger(\gamma^0)^2 e^{-i\alpha\gamma_5}(\gamma^0)e^{i\alpha\gamma_5}\psi = \bar\psi e^{2i\alpha\gamma_5}\psi = \bar\psi\psi\cos 2\alpha + i\bar\psi\gamma_5\psi\sin 2\alpha \\ \bar\psi\gamma_5\psi &\to \psi^\dagger(\gamma^0)^2 e^{-i\alpha\gamma_5}(\gamma^0)\gamma_5 e^{i\alpha\gamma_5}\psi = \bar\psi e^{2i\alpha\gamma_5}\gamma_5\psi = i\bar\psi\psi\sin 2\alpha + \bar\psi\gamma_5\psi\cos 2\alpha.\end{aligned}$$

For any nonzero angle α, the bilinears $\bar\psi\psi$ and $\bar\psi\gamma_5\psi$ transform into linear combinations of each other.

(d) Consider a general quantum field theory in d dimensions with action $S[\phi]$ (for multiple fields pretend that ϕ is a column vector of fields). Saying that it has a "symmetry" $U : \phi \to \phi'$ means much more than just the action is invariant, $S[\phi] = S[\phi']$. In fact, $\hat U$ should be a unitary operator that commutes with the Hamiltonian so it should be possible to label states by a nonnegative energy and a pure phase. If the transformation is parametrized by a continuous variable, θ, then there should exist, by Noether's theorem, a conserved current. The spatial integral of the time component of the current yields a charge identified as the eigenvalue of a Hermitian operator $\hat Q$. Then $\hat U(\theta) = e^{i\theta\hat Q}$. The operator statement of charge conservation is $d\hat Q/dt = i[\hat H, \hat Q] = 0$. If we define the action of a symmetry transformation on an arbitrary operator as $\hat{\mathcal{O}}' = \hat U(\theta)\hat{\mathcal{O}}\hat U(\theta)^\dagger$, then for infinitesimal θ, $\delta_\theta\hat{\mathcal{O}} = i\theta[\hat Q, \hat{\mathcal{O}}]$. If we choose $\hat{\mathcal{O}}$ to be a field operator, then this means

that the conserved charge generates the corresponding field transformation. That is, $\hat{Q}$ is given by a spatial integral of a charge density in such a way that applying the canonical commutation relations gives the infinitesimal form of the transformation law for the field.

Let us derive Noether's theorem for the symmetry transformation $\phi'(x) = \phi(x) + \epsilon\delta\phi(x)$, where ϵ is infinitesimal. Consider the Euclidean functional integral $Z = \int \mathcal{D}\phi\, e^{-S[\phi]}$. A symmetry transformation leaves both the action and measure invariant, $\mathcal{D}\phi'\, e^{-S[\phi']} = \mathcal{D}\phi\, e^{-S[\phi]}$. Now consider the transformation given by $\phi'(x) = \phi(x) + \epsilon\rho(x)\delta\phi(x)$, where for the moment ρ is an arbitrary function. The path integral measure times weight is only invariant under this transformation if ρ is constant. Therefore, the variation must be proportional to the gradient of ρ, $\mathcal{D}\phi'\, e^{-S[\phi']} = \mathcal{D}\phi\, e^{-S[\phi]} \left[1 + \epsilon \int d^dx\, j^\mu \partial_\mu \rho + O(\epsilon^2)\right]$. The current $j^\mu(x)$ is a local function of the fields and their derivatives that comes from varying the action and measure. Restrict ρ to have finite support in some small region R. We may consider a functional integral with local operator insertions '$\dots$' outside R. The transformation rule does not affect the insertions. A functional integral is, of course, invariant under a change of variables. So

$$0 = \int \mathcal{D}\phi'\, e^{-S[\phi']} \dots - \int \mathcal{D}\phi\, e^{-S[\phi]} \dots = \epsilon \int d^dx\, \rho(x) \langle \partial_\mu j^\mu \dots \rangle + O(\epsilon^2).$$

Note that an integration by parts did not affect the local insertions because they are outside R. Since ρ is arbitrary it follows that $\langle \partial_\mu j^\mu \dots \rangle = 0$. This means that $\partial_\mu \hat{j}^\mu = 0$ holds as a local operator equation for points in space such that the gradient of the current is not coincident with some other operator.

For a four-dimensional theory of a free massless Dirac fermion the action $S = \int d^4x\, \bar{\psi}\partial\!\!\!/\psi$ is invariant under continuous chiral transformations. It remains to determine whether we can define the functional integral measure $\mathcal{D}\bar{\psi}\mathcal{D}\psi$ so that it also remains invariant. Following Fujikawa,[1] decompose the fields in a complete basis of orthonormal eigenfunctions of the Hermitian operator $i\partial\!\!\!/$: $\psi(x) = \sum_n a_n \varphi_n(x)$ and $\bar{\psi}(x) = \sum_n \bar{b}_n \varphi_n(x)^\dagger$, where $i\partial\!\!\!/\varphi_n(x) = \lambda_n \varphi(x)$ for real λ_n, $\int d^4x\, \varphi_n(x)^\dagger \varphi_m(x) = \delta_{nm}$, and $\{a_n, \bar{b}_n\}$ are generators of a complex Grassmann algebra. A discrete spectrum is obtained by defining the theory in a four-dimensional box with periodic boundary conditions in all directions. Let $\mathcal{D}\bar{\psi}\mathcal{D}\psi \equiv \prod_n d\bar{b}_n da_n \equiv \lim_{N\to\infty} d\bar{b}_N \cdots d\bar{b}_1 da_N \cdots da_1$. Now consider a chiral transformation $\psi'(x) = e^{i\alpha\gamma_5}\psi(x)$, $\bar{\psi}'(x) = \bar{\psi}(x) e^{i\alpha\gamma_5}$. The coefficients change as $a'_n = \sum_m c_{nm} a_m$ where $c_{nm} = \int d^4x\, \varphi_n(x)^\dagger e^{i\alpha\gamma_5} \varphi_m(x)$. Recall that Grassmann integration is defined to be a translation invariant linear operation. It is in fact the same operation as partial differentiation. Since partial derivatives anticommute, $\prod_n da'_n = \prod_n \frac{\partial}{\partial a'_n} = \prod_n \sum_m \frac{\partial}{\partial a_m} \frac{\partial a_m}{\partial a'_n} = (\det C)^{-1} \prod_n da_n$, where $C = \|c_{nm}\|$ with $c_{nm} = \partial a'_n / \partial a_m$. So there is potentially a nontrivial Jacobian in the functional integral under a chiral transformation. For infinitesimal α, $c_{nm} = \delta_{nm} + i\alpha \int d^4x\, \varphi_n(x)^\dagger \gamma_5 \varphi_m(x) + O(\alpha^2)$. Therefore, $(\det C)^{-1} = e^{-\mathrm{tr}\ln C} = e^{-i\alpha \int d^4x \sum_n \varphi_n(x)^\dagger \gamma_5 \varphi_n(x) + O(\alpha^2)}$. Define $A(x) \equiv \sum_n \varphi_n(x)^\dagger \gamma_5 \varphi_n(x)$. The Jacobian for the $\mathcal{D}\bar{\psi}$ part of the measure also contributes an identical factor. Thus,

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi \to \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{-2i\alpha \int d^4x\, A(x)}.$$

[1] K. Fujikawa, Phys. Rev. Lett. **42** 1195 (1979). We are studying the somewhat trivial case of a theory without gauge fields. The following discussion and results are much more interesting and subtle when gauge invariance is required.

Clearly, if the density $A(x)$ integrates to some nonzero number, then the measure is not invariant under infinitesimal chiral transformations.

To evaluate $A(x)$ it is important to notice that $[i\partial\!\!\!/, \gamma_5] \neq 0$ so one cannot work with spinors that are simultaneously eigenfunctions of the Dirac operator and the chirality matrix. However, suppose that φ_n has nonzero eigenvalue λ_n under $i\partial\!\!\!/$. Then $\gamma_5\varphi_n$ has a different eigenvalue $-\lambda_n$ due to the anticommutation property $\{\gamma_5, \gamma^\mu\} = 0$. Therefore, φ_n and $\gamma_5\varphi_n$ are linearly independent eigenfunctions and this means that they are orthogonal, $\int d^4x\, \varphi_n(x)^\dagger\gamma_5\varphi_n(x) = 0$. Thus, it is only necessary to consider the contribution to $A(x)$ from the eigenfunctions with zero eigenvalue, often called the "zero modes." Since $i\partial\!\!\!/$ is translationally invariant and we have imposed periodic boundary conditions, the zero modes are simply constant spinors. We take the four linearly independent zero modes to be $(\varphi_n^{\text{zero}})_\beta = \delta_{n\beta}/\sqrt{V}$ (β is a Dirac index and V is the spacetime volume). Thus, $A(x) = \frac{1}{V}\sum_{n=1}^4(\varphi_n^{\text{zero}})^\dagger\gamma_5\varphi_n^{\text{zero}} = \frac{1}{V}\text{tr}\,\gamma_5 = 0$. Unfortunately, this calculation lacks rigor because the evaluation of $A(x)$ can always be done in a different basis. For example, if we choose the basis where γ_5 is diagonal (by 'diagonal' we mean in the infinite dimensional function space that $i\partial\!\!\!/$ lives in) then we will have to sum an infinite number of 1's and -1's to compute $A(x)$. Such a sum is conditionally convergent. A way around this difficulty is to regularize the large eigenvalues (those λ_n whose magnitudes are of order M or larger, where M is some arbitrary mass scale) by inserting a Gaussian damping factor and taking the regulator mass to infinity at the end, $A(x) \equiv \lim_{M\to\infty}\sum_n \varphi_n(x)^\dagger\gamma_5 e^{-(\lambda_n/M)^2}\varphi_n(x)$. Replace the eigenvalue by $i\partial\!\!\!/$. We are allowed to compute this scalar quantity in a different basis. Choose the basis of plane waves (one per direction in Dirac space) and instead of choosing an explicit basis for gamma matrices leave it general by writing the trace over Dirac indices,

$$\begin{aligned} A(x) &= \lim_{M\to\infty}\text{tr}\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\gamma_5\exp[-(i\partial\!\!\!//M)^2]e^{ik\cdot x} \\ &= \lim_{M\to\infty}(\text{tr}\,\gamma_5)\int\frac{d^4k}{(2\pi)^4}e^{-k^2/M^2} = 0 \end{aligned}$$

since γ_5 is traceless.

We have established that the theory has continuous chiral symmetry. The associated conserved charge is given entirely by the variation of the action since $e^{-2i\alpha\int d^4x\,\rho(x)A(x)} = 1$ for arbitrary function ρ. It is straightforward to find $j^\mu = i\bar\psi\gamma_5\gamma^\mu\psi$. It is Hermitian. Let $\hat Q \equiv \int d^3x\,\hat j^0 = \int d^3x\,\hat\psi^\dagger(\vec x)\gamma_5\hat\psi(\vec x)$. Inserting the standard mode expansion of a massless Dirac field, $\hat\psi(\vec x) = \int\frac{d^3p}{(2\pi)^3}(2|\vec p|)^{-1/2}\sum_s\left[\hat b_s(\vec p)u_s(\vec p)e^{i\vec p\cdot\vec x} + \hat d_s(\vec p)^\dagger v_s(\vec p)e^{-i\vec p\cdot\vec x}\right]$,

$$\hat Q = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2|\vec p|}\sum_{s,s'}\left[u_s^\dagger(\vec p)\hat b_s^\dagger(\vec p) + v_s^\dagger(-\vec p)\hat d_s(-\vec p)\right]\gamma_5\left[u_{s'}(\vec p)\hat b_{s'}(\vec p) + \hat d_{s'}^\dagger(-\vec p)v_{s'}(-\vec p)\right].$$

An explicit realization for the u and v spinors is $u_s(\vec p) = \sqrt{|\vec p|}\left(\begin{smallmatrix}\chi_s(\vec p)\\ s\chi_s(\vec p)\end{smallmatrix}\right)$ and $v_s(-\vec p) = \sqrt{|\vec p|}\left(\begin{smallmatrix}-\chi_{-s}(-\vec p)\\ s\chi_{-s}(-\vec p)\end{smallmatrix}\right)$ where $s = \pm1$ and $\chi_\pm(\vec p)$ is a two-component basis spinor of eigenvalue ±1 under the helicity operator $\vec\sigma\cdot\vec p/|\vec p|$. Notice that $\chi_\pm^\dagger(\vec p)\chi_\pm(-\vec p) = 0$. Choose the normalization $\chi_\pm^\dagger(\vec p)\chi_\pm(\vec p) = 1$. Then you can check that the standard normalization and orthogonality conditions are obeyed: $u_s^\dagger(\vec p)u_{s'}(\vec p) = v_s^\dagger(-\vec p)v_{s'}(-\vec p) = 2|\vec p|\delta_{ss'}$ and $u_s^\dagger(\vec p)v_{s'}(-\vec p) = 0$.

The u and v spinors are eigenvectors of the Hermitian matrix $h(\vec{p}) = \vec{\alpha} \cdot \vec{p}$ (with eigenvalues $+|\vec{p}|$ and $-|\vec{p}|$, respectively) which appears when the Hamiltonian is expressed in second quantized form in terms of spinor fields. In terms of Dirac's matrices, the standard representation for gamma matrices is $\gamma^0_{\text{Eucl.}} = \beta$ and $\vec{\gamma} = -i\beta\vec{\alpha}$. Then $\gamma_5 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Plugging in we find that $u_s^\dagger(\vec{p})\gamma_5 u_{s'}(\vec{p}) = 2|\vec{p}|s\delta_{ss'}$, $u_s^\dagger(\vec{p})\gamma_5 v_{s'}(-\vec{p}) = 0$, $v_s^\dagger(-\vec{p})\gamma_5 u_{s'}(\vec{p}) = 0$, and $v_s^\dagger(-\vec{p})\gamma_5 v_{s'}(-\vec{p}) = -2|\vec{p}|s\delta_{ss'}$. Using the canonical anticommutation relation to interchange $\hat{d}_s$ and $\hat{d}_s^\dagger$ and changing $\vec{p} \to -\vec{p}$ for these creation and annihilation operators yields

$$\hat{Q} = \int \frac{d^3p}{(2\pi)^3} \sum_{s=\pm} s \left[\hat{b}_s^\dagger(\vec{p})\hat{b}_s(\vec{p}) + \hat{d}_s^\dagger(\vec{p})\hat{d}_s(\vec{p}) \right].$$

Particles are created by $\hat{b}_s^\dagger$ and antiparticles are created by $\hat{d}_s^\dagger$. Therefore the charge counts the number of particles and antiparticles with positive helicity minus the number of particles and antiparticles with negative helicity. In other words, $\hat{Q}$ measures the net helicity.

It is worth mentioning that $\partial_\mu \hat{j}^\mu \neq 0$ when a photon field is coupled to the massless Dirac fermion simply because continuous chiral transformations are no longer a symmetry of the theory. One can in fact understand why $\hat{Q}$ is not conserved by studying a simple example of a single electron in a background vector potential. By solving the Dirac equation for the lowest energy levels when an adiabatically changing electric field is produced parallel to a constant magnetic field B, one finds that Q changes by an amount proportional to eB.

(e) If the Lagrange density is $\mathcal{L} = \bar{\psi}(\partial\!\!\!/ + m + i\tilde{m}\gamma_5)\psi$, then the Hamiltonian density is $h(\vec{x}) = \bar{\psi}(\vec{x})(\vec{\gamma} \cdot \vec{\nabla} + m + i\tilde{m}\gamma_5)\psi(\vec{x})$. This needs to be Hermitian. Since $i\bar{\psi}\gamma_5\psi$ is Hermitian ($\{\gamma^0, \gamma_5\} = 0$ gives one minus sign, but $i^* = -i$ gives another), $\tilde{m}$ must be real. The Grassmann functional integral defining this theory is Gaussian so you should expect to be able to solve it exactly. Indeed, we may make a chiral transformation that puts $\mathcal{L}$ into a form from which we can simply read off the answer. Using part (c),

$$\mathcal{L} \to \bar{\psi}\partial\!\!\!/\psi + \bar{\psi}\psi(m\cos 2\alpha - \tilde{m}\sin 2\alpha) + i\bar{\psi}\gamma_5\psi(m\sin 2\alpha + \tilde{m}\cos 2\alpha).$$

Choose α such that the coefficient of the $i\bar{\psi}\gamma_5\psi$ bilinear vanishes. That is, find the value of α in the interval $(-\pi/2, \pi/2)$ that satisfies $\tan 2\alpha = -\tilde{m}/m$. Some simple trignometry gives $\cos 2\alpha = m/\sqrt{m^2 + \tilde{m}^2}$. Then it is straightforward to find that the coefficient of the $\bar{\psi}\psi$ bilinear is $m_{\text{phy}} \equiv \sqrt{m^2 + \tilde{m}^2}$. The Lagrange density is now just $\mathcal{L} = \bar{\psi}(\partial\!\!\!/ + m_{\text{phy}})\psi$. Keep in mind that these spinor fields are different from the original ones. In terms of a functional integral, we have made a change of integration variables. The time-ordered correlator is simply $i\langle 0|\mathcal{T}(\psi(x)\bar{\psi}(y))|0\rangle = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)}(ip\!\!\!/ + m_{\text{phy}})^{-1}$. The physical particle mass is given by the location of the pole, $\sqrt{-p^2} = m_{\text{phy}}$.

2. (a) Consider a Euclidean theory of a real scalar and Dirac fermion in $d = 4$ with Lagrange density $\mathcal{L}(x) = \sum_i c_i \mathcal{O}_i(x)$. Renormalizability requires that no operator comes with a coupling that has negative mass dimension. In other words, we need the canonical dimension of each operator $\mathcal{O}_i$ to satisfy $d_i \leq 4$. We impose invariance under Lorentz transformations (in Euclidean space this is just a proper four-dimensional rotation): $\mathcal{O}_i(x) = \mathcal{O}_i(\Lambda x)$, continuous phase rotation of the fermions ($U(1)_F$): $\mathcal{O}_i[\bar{\psi}, \psi] = \mathcal{O}_i[\bar{\psi}e^{-i\alpha}, e^{i\alpha}\psi]$, and discrete chiral transformations ($\mathbb{Z}_2^{\text{chiral}}$): $\mathcal{O}_i[\phi, \bar{\psi}, \psi] = \mathcal{O}_i[-\phi, -\bar{\psi}\gamma_5, \gamma_5\psi]$. At $d_i = 0$ there is only the unit operator. At $d_i = 1$ there is nothing; in particular, ϕ is odd under $\mathbb{Z}_2^{\text{chiral}}$. At $d_i = 2$ we have only ϕ^2. At $d_i = 3$ there is nothing; in particular, both ϕ^3 and $\bar{\psi}\psi$ are odd under $\mathbb{Z}_2^{\text{chiral}}$. At $d_i = 4$ we have the usual kinetic terms $(\partial\phi)^2$, $\bar{\psi}\partial\!\!\!/\psi$ and the interactions ϕ^4, $\phi\bar{\psi}\psi$. We conclude that the most general renormalizable Yukawa theory with symmetries Poincaré + $U(1)_F$ + $\mathbb{Z}_2^{\text{chiral}}$ has a Lagrange density, defined at the momentum scale Λ,

$$\mathcal{L} = \frac{1}{2}Z_\phi(\Lambda)(\partial\phi)^2 + \frac{1}{2}m^2(\Lambda)\phi^2 + \frac{\lambda(\Lambda)}{4!}\phi^4 + Z_\psi(\Lambda)\bar{\psi}\partial\!\!\!/\psi + g(\Lambda)\phi\bar{\psi}\psi + c(\Lambda).$$

To demonstrate perturbative renormalizability in a systematic way we need to calculate connected correlation functions starting with the two-point function. When we come across a 1PI diagram that is superficially divergent (i.e., it has more powers of loop momenta in the numerator than in the denominator), then we must cancel the UV cutoff sensitivity by suitably adjusting the couplings, or equivalently, adding counterterms. Once all superficially divergent diagrams have been taken care of, it is guaranteed that any higher correlation function may be computed and renormalized (i.e., made finite) without introducing new counterterms. To be more precise, in a Feynman integral let the superficial degree of divergence D be the difference in number of momenta in the numerator (coming from loop integrals and derivative interactions) and the number of momenta in the denominator (arising from propagators).[2] For a connected diagram in our Yukawa theory, $D = 4 - B - (3/2)F$, where B is the number of external boson lines and F is the number of external fermion lines. Let us list the possible 1PI diagrams with $D \geq 0$ that contribute to an n-point function. Starting at $n = 2$ we can have $(B, F) = (2, 0)$ or $(0, 2)$. At $n = 3$ we can have $(3, 0)$ or $(1, 2)$. Finally, for $n = 4$ we have just $(4, 0)$. For $n \geq 5$ we will always have $D < 0$. This means it is only necessary to compute the two-, three-, and four-point functions to work out the renormalization of the theory. Indeed, you should have expected that m^2, g, and λ will need to be renormalized when computing precisely these correlators.

(i) If $\mathbb{Z}_2^{\text{chiral}}$ was not a symmetry, then we could include the terms ϕ^3, $\bar{\psi}\psi$, $i\bar{\psi}\gamma_5\psi$. Note that a linear term ϕ can always be removed by shifting the scalar field, it is equivalent to allowing a cubic term.

(ii) A $U(1)_F$-variant operator needs to have the structure $\psi^T A\psi$ where A is some four-by-four matrix that is chosen to preserve Lorentz invariance and commutes with γ_5. Let us try to ensure invariance under proper Lorentz transformations $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ first.[3] Recall that the spinor field transforms as $\psi(x) \to S\psi(x')$ where $S = \exp(\frac{1}{8}\omega_{\mu\nu}[\gamma^\mu, \gamma^\nu])$ and $\omega^\mu{}_\nu = \Lambda^\mu{}_\nu - \delta^\mu{}_\nu$. Therefore, $\psi^T A\psi \to \psi^T S^T A S\psi$. Now for infinitesimal ω, $S^T =$

[2] See Coleman, *Aspects of Symmetry*, Ch. 4.

[3] For this problem we are switching back to Minkowski space where γ^0 is anti-Hermitian and squares to -1.

$1 + \frac{1}{8}\omega_{\mu\nu}[(\gamma^\nu)^T, (\gamma^\mu)^T] + O(\omega^2)$. There is a nice relation between gamma matrices and their transposes. Recall that one can find a real, symmetric "charge conjugation" matrix C such that $C = C^{-1}$ and $C\vec{\alpha}^T C = \vec{\alpha}$, $C\beta^T C = -\beta$ where β, $\vec{\alpha}$ are Dirac's matrices in the standard representation. From the definitions $\gamma^0 = -i\beta$, $\vec{\gamma} = -i\beta\vec{\alpha}$ one can easily find $(\gamma^\mu)^T = C\gamma^0\gamma^\mu\gamma^0 C$. This allows us to write $S^T = 1 + \frac{1}{8}\omega_{\mu\nu}C\gamma^0[\gamma^\mu, \gamma^\nu]\gamma^0 C + O(\omega^2) = -C\gamma^0(1 - \frac{1}{8}\omega_{\mu\nu}[\gamma^\mu, \gamma^\nu] + O(\omega^2))\gamma^0 C = -C\gamma^0 S^{-1}\gamma^0 C$. So we need to find an A that satisfies the condition $-C\gamma^0 S^{-1}\gamma^0 CAS = A$. One can easily check that $A = Ci\gamma^0$ works. Moreover, A commutes with γ_5 since $C = \gamma^2$. Hence, the sought-after operator is $\psi^T Ci\gamma^0\psi$. It is natural to define a charge-conjugated spinor $\psi_C \equiv C\psi^* = C(\psi^\dagger)^T$ (if ψ were a Majorana field then $\psi_C = \psi$). Note that $\psi_C^\dagger = \psi^T C$. So one can write the operator as $\bar{\psi}_C\psi$ which looks like an "ordinary" Dirac mass term except that one field is replaced by its charge-conjugate. We need a Hermitian operator so take $\bar{\psi}_C\psi + \bar{\psi}\psi_C$.

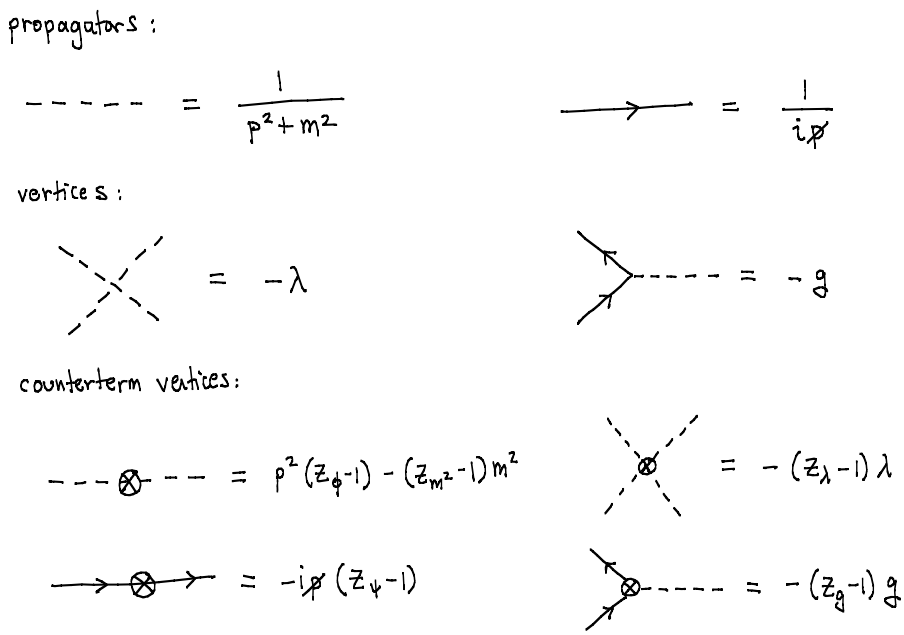
(iii) If parity was not a symmetry, then we could include the pseudoscalar $i\phi\bar{\psi}\gamma_5\psi$. If we do not require proper Lorentz invariance, then we are free to introduce tensors such as $\partial_\mu\phi\partial_\nu\phi$, $\psi^\dagger\psi$, $\bar{\psi}\gamma^\mu\psi$, $\bar{\psi}\gamma_5\gamma^\mu\psi$, $i\phi\bar{\psi}[\gamma^\mu, \gamma^\nu]\psi$.

(b) Our Yukawa theory is defined at the momentum scale Λ. Assume that $m^2(\Lambda) > 0$ so that the saddle point is given by $\phi = 0$.

For doing perturbation theory at a momentum scale $\mu < \Lambda$ it is trivial, yet quite convenient, to write the field rescalings as $Z_{\text{field}} = 1 + (Z_{\text{field}} - 1)$. The "1" leads to a canonically normalized kinetic term, while the difference leads to a momentum-dependent counterterm. Let us also write the remaining "bare" coefficients in terms of "renormalized" coefficients

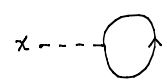
$$\begin{aligned} m^2(\Lambda) &= Z_{m^2}(\Lambda, \mu)m^2(\mu), \\ \lambda(\Lambda) &= Z_\lambda(\Lambda, \mu)\lambda(\mu), \\ g(\Lambda) &= Z_g(\Lambda, \mu)g(\mu), \end{aligned}$$

where these "Z-factors" relate the renormalized coupling at the arbitrary momentum scale μ back to the bare coupling at the original scale Λ. For perturbation theory it is again convenient to make a trivial rewriting $c(\Lambda) = c(\mu) + (Z_c - 1)c(\mu)$. The latter part leads to a counterterm. If a value of μ is chosen such that $\lambda(\mu)$, $g(\mu) \ll 1$ and $\lambda(\mu) \sim O(g(\mu)^2)$, then we may do an asymptotic expansion around free field theory with the renormalized couplings as expansion parameters. The Euclidean space Feynman rules are precisely those discussed by Prof. Yaffe in lectures on Jan. 20 and 22 augmented with the corresponding counterterm interactions. The rules are summarized below (henceforth, when it is not important to the discussion we will supress writing the dependence of the renormalized coupling on μ)

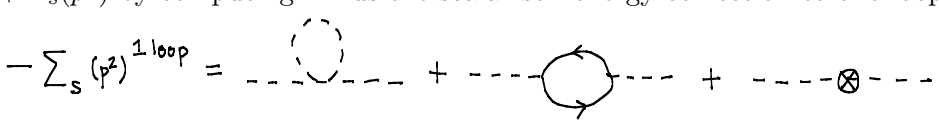


Feynman rules around $\phi = 0$.

First consider the one-point function $\langle 0|\phi(x)|0\rangle$. At one loop there is a tadpole diagram shown at right which equals $-g\int \frac{d^4q}{(2\pi)^4}\mathrm{tr}[1/i\not{q}] = 0$ since the trace of an odd number of gamma matrices vanishes.



Next calculate the (inverse) scalar-scalar correlator, $\left[\int d^4x\, e^{-ip\cdot x}\langle 0|\mathcal{T}(\phi(x)\phi(0))|0\rangle\right]^{-1} = p^2 + m^2 + \Sigma_s(p^2)$ by computing minus the scalar self-energy correction to one loop order,



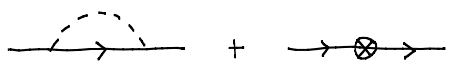
$$
\begin{aligned}
-\Sigma_s(p^2)^{1\text{ loop}} &= -\frac{\lambda}{2}\int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2+m^2} - (-g)^2\int \frac{d^4q}{(2\pi)^4}\mathrm{tr}\left[\frac{1}{i\not{q}}\frac{1}{i(\not{q}+\not{p})}\right] \\
&\quad + p^2(Z_\phi - 1) - (Z_{m^2}-1)m^2 \\
&= -\frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2\ln\left(\frac{\Lambda^2}{m^2}\right)\right] + \frac{g^2}{4\pi^2}\left[\Lambda^2 - \frac{p^2}{2}\ln\left(\frac{\Lambda^2}{p^2}\right)\right] \\
&\quad + p^2(Z_\phi - 1) - (Z_{m^2}-1)m^2 + (\text{finite as } \Lambda\to\infty).
\end{aligned}
$$

Two loop corrections are $O(\lambda^2, g^4, \lambda g^2)$. Let us explain how to calculate the cutoff-sensitive parts of these integrals: for the scalar loop consider $q^2 \gg m^2$ so that you can expand $(q^2+m^2)^{-1} \approx q^{-2}(1 - m^2q^{-2} + O(m^4q^{-4}))$. The first term leads to a quadratically divergent integral, the second leads to a logarithmically divergent integral, and higher order terms give finite integrals. The appropriate "lower limit of integration" for the logarithmically divergent integral is the momentum scale at which the assumption that q^2 is larger than

any other momentum goes wrong. This scale is obviously m^2. Therefore, the log divergence appears as $\ln(\Lambda^2/m^2)$. By the way, you already calculated this integral in assignment #6 of Ph 570. For the fermion loop first rationalize the propagators so that the trace over Dirac structure sits solely in the numerator. Use the identities $\mathrm{tr}\mathbb{1} = 4$ and $\mathrm{tr}(\not{p}\not{q}) = 4p \cdot q$. Again, expand around large q^2 and drop terms which are $O(q^{-5})$ and higher. Keep in mind that d^4q is a rotationally invariant measure (and the cutoff Λ is a rotationally invariant regulator) so any integrand with a single $q \cdot p$ integrates to zero. Also, when you see something like $q_\mu q_\nu$ you can replace it with $(q^2/d)\delta_{\mu\nu}$ inside the integral. You should find integrals that diverge quadratically and logarithmically. This technique of considering just the UV behavior of loop integrals makes it very difficult to calculate precisely the finite terms, but we are ignoring them anyway.

We want to define the theory, at least perturbatively, for arbitrarily large momentum p in the Fourier transform of the correlator. Taking $\Lambda \to \infty$ causes the correlator to diverge. We can cancel this cutoff sensitivity by choosing $(Z_{m^2} - 1)m^2 = -\frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2 \ln\left(\frac{\Lambda^2}{\mu^2}\right)\right] + \frac{g^2}{4\pi^2}\Lambda^2$ and $Z_\phi - 1 = \frac{g^2}{8\pi^2}\ln\left(\frac{\Lambda^2}{\mu^2}\right)$. This choice yields a finite self-energy. Note that the wavefunction rescaling counterterm does not cancel the entire log, it leaves behind a term proportional to $g^2p^2\ln(p^2/\mu^2)$.

The (inverse) fermion-fermion correlator is $\left[\int d^4x\, e^{-ip\cdot x}\langle 0|\mathcal{T}(\bar{\psi}(x)\psi(0))|0\rangle\right]^{-1} = i\not{p} + \Sigma_f(p)$, where minus the fermion self-energy correction to one loop order is given by

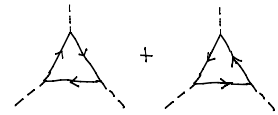
$-\Sigma_f(p)^{1\,\mathrm{loop}} =$ 

$$
\begin{aligned}
-\Sigma_f(p)^{1\text{ loop}} &= (-g)^2 \int \frac{d^4q}{(2\pi)^4}\frac{1}{i(\not{q}+\not{p})}\frac{1}{q^2+m^2} - i\not{p}(Z_\psi - 1) \\
&= -i\not{p}\frac{g^2}{32\pi^2}\ln\left(\frac{\Lambda^2}{\max(p^2,m^2)}\right) - i\not{p}(Z_\psi - 1).
\end{aligned}
$$

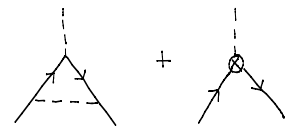
Two loop corrections are $O(g^4, \lambda g^2)$. In general, the fermion self-energy is a four-by-four matrix in Dirac spin space, $\Sigma_f(p) = am + b\not{p}$. However, chiral symmetry forbids the generation of a bilinear $m_{\text{eff}}\bar{\psi}\psi$ in the effective action. Therefore, a must be zero to all orders in perturbation theory. We indeed find that $a = 0$ at one loop order. To understand the cutoff sensitivity it is again only necessary to evaluate the integral for large q. You will need to use the fact that $\int^\Lambda d^4q\,\not{q} = 0$. The scale at which the large q expansion breaks down is the larger of p and m. Nevertheless, it does not matter which scale sits in the log as $\ln(\Lambda^2/p^2) = \ln(\Lambda^2/m^2) + (\text{finite as } \Lambda \to \infty)$. In order to remove the cutoff sensitivity we take $Z_\psi - 1 = -\frac{g^2}{32\pi^2}\ln\left(\frac{\Lambda^2}{\mu^2}\right)$.

For higher-point functions it is only necessary to focus on the interaction vertex. The legs, which have already been renormalized by considering the two-point functions, provide nothing new to the analysis. Therefore, we only need to consider amputated diagrams.

Consider the three-point function $\langle 0|\mathcal{T}(\phi(x)\phi(y)\phi(z))|0\rangle$. This is zero. It must be because it is odd under the chiral transformation. At the one loop level you need to evaluate the diagrams shown at right. The fermion loop involves the trace of three gamma matrices which vanishes identically. If the fermion had a mass, then certain parts of the trace would not be zero and the resulting integrals could be UV sensitive. Again, our lack of a fermion mass term is required by the chiral symmetry so the theory is self-consistent.



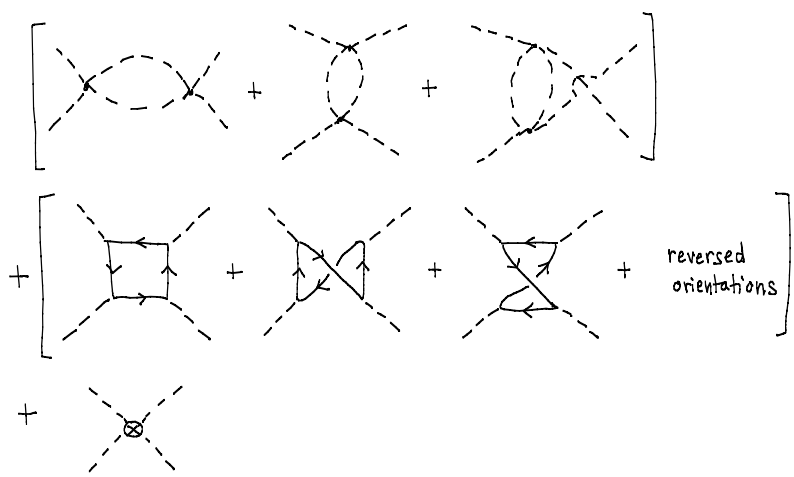
Consider the three-point function $\langle 0|\mathcal{T}(\phi(x)\bar{\psi}(y)\psi(z))|0\rangle$. The one loop contribution is shown at right and evaluates to, dropping all terms which are finite as $\Lambda \to \infty$,



$$
\begin{aligned}
&(-g)^3 \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2+m^2} \frac{1}{i(\not{q}+\not{k}_1)} \frac{1}{i(\not{q}-\not{k}_2)} - (Z_g-1)g \\
\sim &\frac{g^3}{16\pi^4} \int^{\Lambda} \frac{d^4q}{q^4} - (Z_g-1)g \\
\sim &\frac{g^3}{16\pi^2} \ln\left(\frac{\Lambda^2}{\max(m^2,k_1^2,k_2^2)}\right) - (Z_g-1)g.
\end{aligned}
$$

The cutoff sensitivity is removed by choosing $(Z_g-1)g = \frac{g^3}{16\pi^2}\ln\left(\frac{\Lambda^2}{\mu^2}\right)$. The two loop correction is $O(g^5, g^3\lambda)$.

Finally, consider the four-point function $\langle 0|\mathcal{T}(\phi(x)\phi(y)\phi(z)\phi(w))|0\rangle$. Diagrammatically, the one loop contribution is



There are nine one loop diagrams. As you can see there are really just two basic diagrams

and the rest are permutations of the external momenta. They evaluate to

$$\frac{1}{2}(-\lambda)^2\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2+m^2}\frac{1}{(q+k_1+k_2)^2+m^2}+(3\text{ perms})$$
$$-(-g)^4\int\frac{d^4q}{(2\pi)^4}\mathrm{tr}\left[\frac{1}{i\not{q}}\frac{1}{i(\not{q}+\not{k}_1)}\frac{1}{i(\not{q}+\not{k}_1+\not{k}_2)}\frac{1}{i(\not{q}+\not{k}_1+\not{k}_2+\not{k}_3)}\right]+(5\text{ perms})-(Z_\lambda-1)\lambda$$
$$\sim\frac{3\lambda^2}{32\pi^2}\ln\left(\frac{\Lambda^2}{\max(m^2,k_i^2)}\right)-\frac{3g^4}{2\pi^2}\ln\left(\frac{\Lambda^2}{\max(m^2,k_i^2)}\right)-(Z_\lambda-1)\lambda.$$

The cutoff sensitivity is removed by choosing $(Z_\lambda-1)\lambda=\frac{3\lambda^2-48g^4}{32\pi^2}\ln\left(\frac{\Lambda^2}{\mu^2}\right)$. The two loop correction is $O(\lambda^3,g^6,\lambda g^4,\lambda^2g^2)$.

Summarizing, to one loop order the Z-factors are

$$\begin{aligned}
Z_\phi &= 1+\frac{g^2}{8\pi^2}\ln(\Lambda^2/\mu^2)\\
Z_\psi &= 1-\frac{g^2}{32\pi^2}\ln(\Lambda^2/\mu^2)\\
Z_{m^2} &= 1+\frac{8g^2-\lambda}{32\pi^2}\frac{\Lambda^2}{m^2}+\frac{\lambda}{32\pi^2}\ln(\Lambda^2/\mu^2)\\
Z_g &= 1+\frac{g^2}{16\pi^2}\ln(\Lambda^2/\mu^2)\\
Z_\lambda &= 1+\frac{3\lambda-48g^4/\lambda}{32\pi^2}\ln(\Lambda^2/\mu^2).
\end{aligned}$$

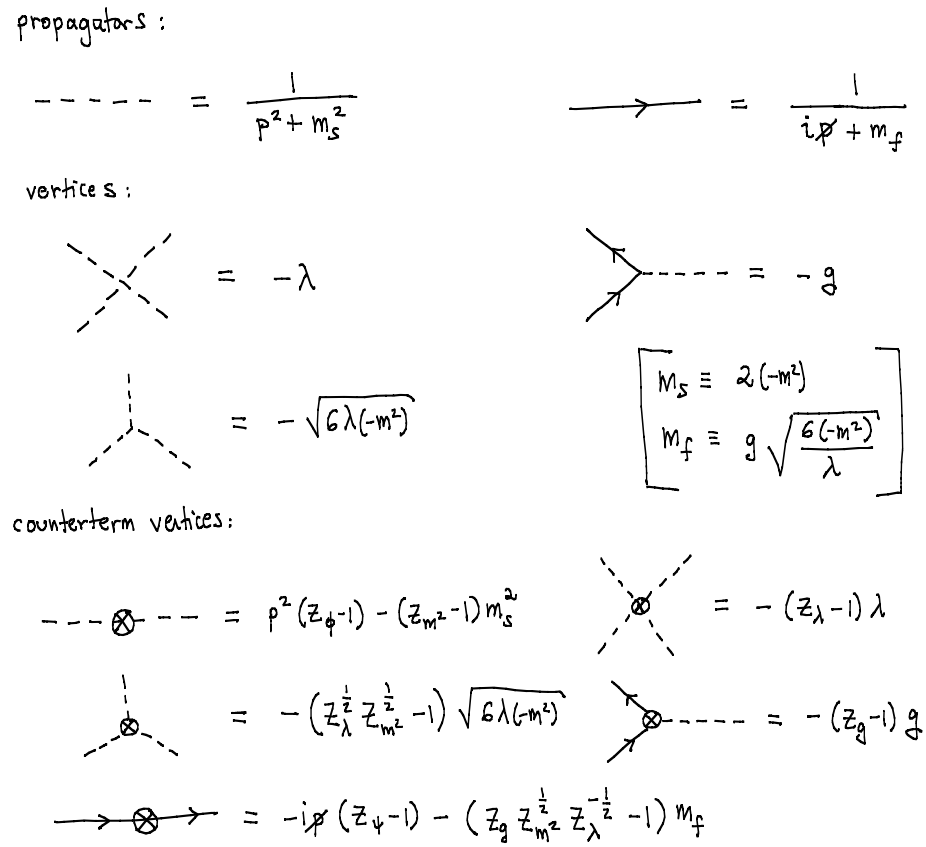
The one loop renormalization group equations for the couplings are[4] $\beta_g\equiv\mu\frac{dg(\mu)}{d\mu}\approx\frac{g(\mu)^2}{8\pi^2}$ and $\beta_\lambda\equiv\mu\frac{d\lambda(\mu)}{d\mu}\approx\frac{3\lambda(\mu)^2-48g(\mu)^4}{16\pi^2}$.

(c) If $m^2(\Lambda)<0$, then the saddle point is given by $\phi=\pm\sqrt{\frac{6|m^2(\Lambda)|}{\lambda(\Lambda)}}\equiv\pm v(\Lambda)$. We may choose either minimum and expand around it as $\phi(x)=v+\sigma(x)$. The Lagrange density in terms of the shifted scalar field takes the form

$$\begin{aligned}
\mathcal{L}=&\frac{1}{2}Z_\phi(\Lambda)(\partial\sigma)^2+(-2m^2(\Lambda))\frac{\sigma^2}{2}+\sqrt{-6\lambda(\Lambda)m^2(\Lambda)}\frac{\sigma^3}{3!}+\lambda(\Lambda)\frac{\sigma^4}{4!}\\
&+Z_\psi(\Lambda)\bar\psi\not{\partial}\psi+g(\Lambda)\sqrt{\frac{-6m^2(\Lambda)}{\lambda(\Lambda)}}\bar\psi\psi+g(\Lambda)\sigma\bar\psi\psi+\tilde{c}(\Lambda).
\end{aligned}$$

By writing $\mathcal{L}$ as an expansion around free field theory we obtain the Euclidean space Feynman rules. Dashed lines represent the σ field and the parameters appearing in the rules are the renormalized (not bare) values.

[4] We are studying the behavior of the renormalized coupling as μ is varied. The key statement is that $\frac{d}{d\mu}$(bare coupling) $=0$. It is also fine to study the behavior of the bare coupling as Λ is varied. The same RG equations will be obtained. The latter point of view is espoused in assignment #6 of Ph 570. There the key statement is that $\frac{d}{d\Lambda}$(renormalized coupling) $=0$.



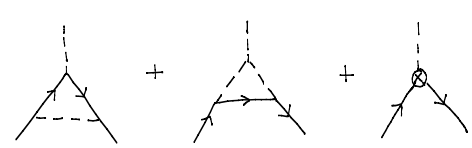
Feynman rules around $\phi = +v$.

The fermion self-energy, at one loop, is represented by the same 1PI diagrams considered earlier for the $m^2(\Lambda) > 0$ case. However, upon evaluating the cutoff sensitive part of the diagram we get a nonzero contribution proportional to the identity matrix in Dirac space,

$$-\Sigma_f(p)^{1\text{ loop}} = (-\frac{i}{2}\not{p} + m_f)\left[\frac{g^2}{16\pi^2}\ln\left(\frac{\Lambda^2}{\max(p^2, m_s^2, m_f^2)}\right)\right] - i\not{p}(Z_\psi - 1) - (Z_g Z_{m^2}^{1/2} Z_\lambda^{-1/2} - 1) m_f.$$

As before, to cancel the cutoff sensitivity we set $Z_\psi - 1 = -\frac{g^2}{32\pi^2}\ln\left(\frac{\Lambda^2}{\mu^2}\right)$, but now also $Z_g Z_{m^2}^{1/2} Z_\lambda^{-1/2} - 1 = \frac{g^2}{16\pi^2}\ln\left(\frac{\Lambda^2}{\mu^2}\right)$. So the fermion-fermion correlator requires both wave-function and mass renormalization. We stress that the counterterm is not new in the sense that it arises from a bare term $m_f(\Lambda)\bar{\psi}\psi$ that has been added to $\mathcal{L}$. Rather, the counterterm comes from the renormalization of the original parameters.

The 1PI, one loop contribution to the three-point function $\langle\phi\bar{\psi}\psi\rangle$ is given diagrammatically at right. The second diagram is new and is made possible by the cubic scalar vertex. Notice that it is not cutoff sensitive. For large loop momenta, the integral scales as $\int^\Lambda d^4q\, q^{-5}$ which converges. As before, only the first diagram has cutoff sensitivity and this is canceled by renormalizing g.



This loop integral diverges logarithmically so the only place m_f can enter is in a logarithm along with Λ. Therefore, $(Z_g - 1)g$ is still equal to $\frac{g^3}{16\pi^2} \ln\left(\frac{\Lambda^2}{\mu^2}\right)$.

Part (c) of this problem is intended to show you that even when a nontrivial minimum of the action is chosen, and the corresponding Feynman rules are markedly different and more complicated, all of the UV sensitivity can still be absorbed by the original couplings. At first this might seem quite miraculous—there is a cubic scalar vertex that wasn't there in the original formulation so there are more 1PI diagrams to consider. A full one loop analysis that, in addition to the above analyses of the fermion self-energy and Yukawa interaction, also includes the scalar self-energy and the scalar three- and four-point functions leads to a set of equations for the Z-factors. However, there will be more equations than unknowns so the choice of Z-factors must self-consistently satisfy all the equations. From this point of view the fact that we can absorb all the divergences into just three couplings seems quite amazing. Of course, a key point is that although there is an extra Feynman rule for the cubic scalar vertex, the rule involves the original couplings in a very particular combination. We know that the action in its original form was constructed to be renormalizable. Rewriting it shouldn't change this basic fact. So from this point of view the renormalizability of the action in terms of a shifted field is mundane.

3. (a) By inserting the defining relation $[t_a, t_b] = i f_{ab}{}^c t_c$ into the identity $[t_a, [t_b, t_c]] + [t_b, [t_c, t_a]] + [t_c, [t_a, t_b]] = 0$ we get $(f_{ab}{}^d f_{cd}{}^e + f_{bc}{}^d f_{ad}{}^e + f_{ca}{}^d f_{bd}{}^e) t_e = 0$. Since the basis elements of a Lie algebra are linearly independent and may always be chosen orthogonal, it follows that the term in parentheses equals zero. This is a Jacobi identity.

Write the defining relation in the form $(t_a^{\mathrm{R}})_{cd}(t_b^{\mathrm{R}})_{de} - (t_b^{\mathrm{R}})_{cd}(t_a^{\mathrm{R}})_{de} = f_{ab}{}^d (t_d^{\mathrm{R}})_{ce}$ and plug in the ansatz $(t_b^{\mathrm{adj}})_{ac} = i f_{ab}{}^c$. This gives $f_{ca}{}^d f_{db}{}^e - f_{cb}{}^d f_{da}{}^e = f_{ab}{}^d f_{cd}{}^e$. Using the fact that the structure constant is antisymmetric in its lower two indices, transpose db in the first term, and transpose cb and da in the second term. The resulting equation is precisely the Jacobi identity. This proves that the structure constants provide a valid representation for the Lie algebra basis.

(b) $D_\mu \psi$ should also be regarded as a dim(R)-dimensional vector. The components are $(D_\mu \psi)_\alpha = \partial_\mu \psi_\alpha + A_\mu^b (t_b^{\mathrm{R}}/i)_{\alpha\gamma} \psi_\gamma$. Here $b = 1, \ldots, \dim(\mathfrak{g})$ and $\alpha, \gamma = 1, \ldots, \dim(\mathrm{R})$.

(c) Put $(t_b^{\mathrm{adj}}/i)_{\alpha\gamma} = f_{\alpha b}{}^\gamma$. Since $\dim(\mathrm{adj}) = \dim(\mathfrak{g})$, Greek and Roman indices run over the same values and it is customary to switch to all Roman indices. Doing so gives $(D_\mu \phi)_a = \partial_\mu \phi_a + f_{ab}{}^c A_\mu^b \phi_c$.

(d) Since the field strength is an element of the Lie algebra, $\mathrm{tr}(F_{\mu\nu}^2) = F_{\mu\nu}^a F_{\mu\nu}^b \mathrm{tr}(t_a^{\mathrm{R}} t_b^{\mathrm{R}})/i^2 = -T_{\mathrm{R}} (F_{\mu\nu}^a)^2$.

(e) We will derive the infinitesimal gauge transformations from the finite ones by expanding $U(x) = e^{-\Lambda(x)}$ to first order in Λ.

(i) From $A_\mu \to U(D_\mu U^{-1})$ we get $\delta A_\mu = \partial_\mu \Lambda + [A_\mu, \Lambda]$. Expand both sides in the Lie algebra basis to get $\delta A_\mu^a = \partial_\mu \Lambda^a + f^{abc} A_\mu^b \Lambda^c$.

(ii) From $F_{\mu\nu} \to U F_{\mu\nu} U^{-1}$ we get $\delta F_{\mu\nu} = [F_{\mu\nu}, \Lambda]$. In components, $\delta F_{\mu\nu}^a = f^{abc} F_{\mu\nu}^b \Lambda^c$.

(iii) From $\phi \to U \phi U^{-1}$ we get $\delta \phi = [\phi, \Lambda]$. In components, $\delta \phi^a = f^{abc} \phi^b \Lambda^c$.

(iv) From $\psi \to D^{\mathrm{R}}(U)\psi$, where $D^{\mathrm{R}}(U) = e^{i\Lambda^a t_a^{\mathrm{R}}}$, we get $\delta \psi_\alpha = i \Lambda^a (t_a^{\mathrm{R}})_{\alpha\beta} \psi_\beta$.

(f) Take the definition of the quadratic Casimir for a representation R of $\mathcal{G}$ and trace: $\mathrm{tr}\left\{\sum_{a=1}^{\dim(\mathcal{G})} t_a^{\mathrm{R}} t_a^{\mathrm{R}} = C_{\mathrm{R}} \mathbb{1}\right\} \Longrightarrow T_{\mathrm{R}} \dim(\mathcal{G}) = C_{\mathrm{R}} \dim(\mathrm{R})$. We remind the reader that the Casimir operator $\sum_a (t_a^{\mathrm{R}})^2$ commutes with all group generators. You are familiar with this operator in the context of the rotation group $SO(3)$: the total angular momentum (squared) $\vec{J}^2$ commutes with J_x, J_y, J_z.

For $\mathcal{G} = SU(N)$ one can count the number of real independent components in an arbitrary $SU(N)$ matrix and find $\dim(\mathcal{G}) = N^2 - 1$. By convention $T_{\mathrm{fund}} = 1/2$ and obviously $\dim(\mathrm{fund}) = N$. Thus, $C_{\mathrm{fund}} = (N^2 - 1)/(2N)$.

For the adjoint representation, recall that $\dim(\mathrm{adj}) = \dim(\mathcal{G})$ (essentially because the Lie algebra and the group have the same dimensionality). Therefore, $C_{\mathrm{adj}} = T_{\mathrm{adj}}$. To find this normalization factor we consider the decomposition of a certain tensor product into irreducible representations, $\mathrm{fund} \otimes \overline{\mathrm{fund}} = \mathrm{singlet} \oplus \mathrm{adj}$. Apply the formula $T_{\mathrm{R}_1 \otimes \mathrm{R}_2} = T_{\mathrm{R}_1} \dim(\mathrm{R}_2) + T_{\mathrm{R}_2} \dim(\mathrm{R}_1)$ on the left and $T_{\mathrm{R}_1 \oplus \mathrm{R}_2} = T_{\mathrm{R}_1} + T_{\mathrm{R}_2}$ on the right.[5] The

[5] To prove the first formula write the generator matrix in the direct product representation as $(t_a^{\mathrm{R}_1 \otimes \mathrm{R}_2})_{iI,jJ} = (t_a^{\mathrm{R}_1})_{ij} \delta_{IJ} + \delta_{ij} (t_a^{\mathrm{R}_2})_{IJ}$, where iI denote row indices and jJ denote column indices.

normalization constant and dimension for the antifundamental representation is the same as the fundamental. The singlet representation has $T_{\text{singlet}} = 0$ because the only generator is zero. Thus, $T_{\text{adj}} = 2T_{\text{fund}} \dim(\text{fund}) = N$.

$i, j = 1, \ldots, \dim(\mathrm{R}_1)$ and $I, J = 1, \ldots, \dim(\mathrm{R}_2)$. Square this matrix and contract with the identity matrix in the direct product representation. The cross terms are proportional to a trace of a single generator which vanishes identically for the group $SU(N)$.

The proof of the second formula is trivial since the trace is a linear operation.