1. (a) We want to transform an arbitrary gauge configuration $A_\mu(x)$ into a gauge equivalent configuration $\bar{A}_\mu(x) \equiv \Omega(x)(\partial_\mu + A_\mu(x))\Omega(x)^\dagger$, with the transformation $\Omega(x)$ chosen so that $\bar{A}_\mu$ satisfies the axial gauge condition $\hat{n}\cdot\bar{A} = 0$. Therefore, we need $\Omega(x)[\hat{n}\cdot\partial + \hat{n}\cdot A(x)]\Omega(x)^\dagger = 0$. For any x, let $x = x_0 + s\hat{n}$, with $s \equiv \hat{n}\cdot x$ and $\hat{n}\cdot x_0 = 0$, and view this condition as a first order differential equation in the variable s. If $A(x)$ were a number (instead of a matrix), then a solution to this first order ODE could be written explicitly as $\Omega(x)^\dagger = \exp\{-\int_0^s ds'\,\hat{n}\cdot A(x_0 + s'\hat{n})\}$. For the general case, where $A(x)$ is a matrix, what we need is a generalization of this exponential defined such that its derivative brings down a factor of $\hat{n}\cdot A(x)$ on the *left* of the exponential. This is just like time-ordering, but in this context is called *path-ordering*. We need $\Omega(x)^\dagger = \mathcal{P}\exp\{-\int_0^s ds'\,\hat{n}\cdot A(x_0 + s'\hat{n})\}$, with $\mathcal{P}$ a path-ordering symbol. Path ordered exponentials may defined by breaking the path into many little pieces, writing the exponential as the product of exponentials of each little contribution to the path, and ordering the resulting factors such that contributions from earlier parts of the path are placed to the right of contributions from later parts.

   The Faddeev-Popov determinant for axial gauge is $\Delta_{\rm FP} = \det(-\hat{n}\cdot D) = \det(-\hat{n}\cdot\partial - [\hat{n}\cdot A, \cdot\,])$. Axial gauge does not uniquely determine the gauge field. One still has the freedom to make gauge transformations by group elements $\Omega(x) = \omega(x^\perp)$ — that is, gauge transformations which have arbitrary dependence on the transverse coordinates $x^\perp$ (transvere to $\hat{n}$), but no dependence on $\hat{n}\cdot x$. In more physical language, in Euclidean space one is free to call the $\hat{n}$ direction the "time" direction. Using such language, the residual gauge invariance amounts to the freedom to make time-independent gauge transformations on fields.

   (b) Transforming to radial gauge is similar to transforming to axial gauge. For any x, let $x = s\hat{x}$, so that $s = |x|$. The required transformation $\Omega(x)$ must satisfy $\Omega(x)[x\cdot\partial + x\cdot A(x)]\Omega(x)^\dagger = 0$, or equivalently $[\frac{d}{ds} + \hat{x}\cdot A(s\hat{x})]\Omega(s\hat{x})^\dagger = 0$. This is again solved by a path ordered exponential, $\Omega(x)^\dagger = \Omega(s\hat{x})^\dagger = \mathcal{P}\exp\{-\int_0^s ds'\,\hat{x}\cdot A(s'\hat{x})\}$.

   (c) Under an infinitesimal gauge transformation, $A_\mu \to A_\mu + D_\mu\Lambda$, where $D_\mu\Lambda \equiv \partial_\mu\Lambda + [A_\mu, \Lambda]$. (Here A_μ is an anti-Hermitian Lie-algebra valued gauge field, and the covariant derivative is an adjoint representation covariant derivative.) Therefore, under such a gauge transformation $\partial\cdot A \to \partial\cdot A + \partial^\mu D_\mu\Lambda$, which implies that the appropriate Faddeev-Popov determinant is $\Delta_{\rm FP} = \det(-\partial^\mu D_\mu) = \det(-\partial^2 - \partial^\mu[A_\mu, \cdot\,])$ (where the last form is just making clear that D_μ is the adjoint representation covariant derivative).

   Using a Gaussian representation of the gauge-fixing delta-function, and expanding the gauge field in a Lie algebra basis, $A_\mu = \sum_a A^a_\mu(T^a/i)$, gives $\int d^dx\,\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 = \int d^dx\,\frac{1}{2}A^a_\mu\left[-\partial^2\delta^{\mu\nu} + (1-\xi^{-1})\partial^\mu\partial^\nu\right]A^a_\nu$ as the quadratic part of the gauge field action. Therefore, the Fourier transformed free gauge field propagator (in Euclidean space) is $D^{(0)}(p)^{ab}_{\mu\nu} = \delta^{ab}\left[\frac{\delta_{\mu\nu}}{p^2} + (\xi-1)\,\frac{p_\mu p_\nu}{p^4}\right]$. Taking the $\xi \to 0$ limit (which was implicit in the Gaussian representation of the gauge fixing delta function) gives $D^{(0)}(p)^{ab}_{\mu\nu} = \delta^{ab}P_{\mu\nu}(p)/p^2$, where $P_{\mu\nu}(p) \equiv \delta_{\mu\nu} - p_\mu p_\nu/p^2$ is a projection operator onto vectors transverse to the momentum.

2. Euclidean space propagators.

   (a) Using $1/p^2 = \int_0^\infty ds\,e^{-sp^2}$, and then changing to $t = x^2/(4s)$ gives $\Delta^{(0)}(x) = \displaystyle\int \frac{d^dp}{(2\pi)^d}\,\frac{e^{ip\cdot x}}{p^2} = \int_0^\infty ds \int \frac{d^dp}{(2\pi)^d}e^{ip\cdot x - sp^2} = \int_0^\infty \frac{ds}{s^{d/2}}\,\frac{e^{-x^2/4s}}{(4\pi)^{d/2}} = \int_0^\infty \frac{dt\,e^{-t}\,t^{(d/2)-2}}{4\pi^{d/2}(x^2)^{(d/2)-1}} = \frac{\Gamma(\frac{d}{2}-1)}{4\pi^{d/2}(x^2)^{(d-2)/2}}$. This reduces to $\Delta^{(0)}(x) = 1/(4\pi^2x^2)$ at $d = 4$, and $\Delta^{(0)}(x) = 1/(4\pi|x|)$ (of course) for $d = 3$.

(b) $\Delta^{(0)}(x;m) = \int \frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot x}}{p^2+m^2} = \int_0^\infty ds \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot x - s(p^2+m^2)} = \frac{1}{(4\pi)^2}\int_0^\infty \frac{ds}{s^2}\, e^{-(sm^2+x^2/4s)}$.
Let $r=\sqrt{x^2}$ and change variables to $t \equiv s(m/r)$ to put the exponent in the canonical form for a saddle point expansion (for large r), $\Delta^{(0)}(x;m) = \frac{m}{(4\pi)^2\, r}\int_0^\infty \frac{dt}{t^2}\, e^{-mrf(t)}$. with $f(t) \equiv t+1/(4t)$. The function $f[t]$ has a unique minimum at $t=\frac12$ with $f(\frac12)=1$ and $f''(\frac12)=4$. So $\Delta^{(0)}(x-y;m) = \frac{4m\, e^{-mrf(1/2)}}{(4\pi)^2\, r}\sqrt{\frac{2\pi}{mrf''(\frac12)}} \times \Big[1+O\Big(\frac{1}{mr}\Big)\Big] = \sqrt{2m}\,\frac{e^{-mr}}{(4\pi r)^{3/2}} \times \Big[1+O\Big(\frac{1}{mr}\Big)\Big]$, with $r \equiv |x-y|$.

(c) $\Delta(x,y;m)$ solves the Green's function equation $(-D[A]^2+m^2)\,\Delta(x,y;m) = \delta^4(x-y)$ with vanishing b.c. at infinity. One may directly analyze this differential equation and show that a solution of the claimed form is the beginning of a systematic expansion in powers of $1/m|x-y|$. But this approach cannot fix the overall normalization of the desired solution. Alternatively, one can construct a path integral representation for $\Delta(x,y;m)$, and then perform an appropriate saddle point expansion to derive the large distance asymptotic behavior. To do so, start with $\Delta(x,y;m) = \langle x|(-D[A]^2+m^2)^{-1}|y\rangle = \int_0^\infty \frac{dT}{2m}\,\langle x|e^{-(T/2m)(-D[A]^2+m^2)}|y\rangle$, and notice that this exponential looks just like the (Euclidean) time evolution operator for a particle in four-dimensional space with Hamiltonian $-D^2/(2m)+m/2$. Therefore, one may derive a path integral representation for matrix elements of this exponential by following the usual procedure of breaking the exponential up into many little steps, leading to $\Delta(x,y;m) = \int_0^\infty \frac{dT}{2m}\int_{z(0)=y}^{z(T)=x} \mathcal{D}z(\tau)\, e^{-S[z]}$ with $S[z] \equiv \int_0^T d\tau\, \big[\frac12 m(dz/d\tau)^2 + A(z(\tau))\cdot(dz/d\tau) + \frac12 m\big]$. Here $z_\mu(\tau)$ is an arbitrary path in Euclidean space which runs from y to x as the parameter τ runs from 0 to T, and $A_\mu(x)$ is the anti-Hermitian gauge field (with $D[A] \equiv \partial + A$). [Note that $\frac12 m\dot z^2 + A(z)\cdot\dot z + \frac12 m$ is the Euclidean Lagrangian corresponding to the Hamiltonian $(p-iA)^2/(2m)+\frac12 m$.] Also note that $e^{-S[z]} = e^{-\frac{m}{2}\int_0^T d\tau\,(\dot z^2+1)}\, U[z]$ where $U[z] \equiv e^{-\int_0^T d\tau\, A(z(\tau))\cdot dz/d\tau} = e^{-\int_y^x A(z)\cdot dz}$ is the exponential of the line integral of $A_\mu(x)$ along the path $z(\tau)$. If the gauge field is non-Abelian, then this exponential needs to be path-ordered.
Since large separation is the same as large mass, *i.e.*, $|x-y| \gg m^{-1}$, we need to find the large mass asymptotic behavior of this path integral. This is easy since the mass only enters through the exponential factor $e^{-\frac{m}{2}\int_0^T d\tau\,(\dot z^2+1)}$. For large mass, this factor is very highly peaked about the path which minimizes $\int_0^T d\tau\,(\dot z^2+1)$ (subject to the boundary conditions of running from y to x in time T). This is just the straight line path $\bar z(\tau)$ which runs from y to x with constant $d\bar z/d\tau = (x-y)/T$. The leading saddle-point approximation to the path integral just gives the result one would have for $A=0$, times the straight line parallel transporter $U[\bar z]$, or $\int_{z(0)=y}^{z(T)=x}\mathcal{D}z(\tau)\, e^{-S[z]} = U[\bar z]\int \frac{d^4p}{(2\pi)^4}\, e^{ip\cdot(x-y)-(T/2m)(p^2+m^2)} = U[\bar z]\,(\frac{m}{2\pi T})^2\, e^{-\frac12 m[T+(x-y)^2/T]}$, with further corrections suppressed by additional powers of $1/m$. The final integral over T, after the change of variables $T=2|x-y|t$, becomes identical to the integral in (b) for the free propagator $\Delta^{(0)}(x-y;m)$. Hence, $\Delta(x,y;m) = \Delta^{(0)}(x-y;m)\, U[\bar z] \times [1+O(1/m|x-y|)]$.

(d) The free propagator $S^{(0)}(x-y)$ for massless fermions is the Green's function for $\not\partial$. Since $(\not\partial)^2 = \partial^2$, $S^{(0)}(x) = -\not\partial\Delta^{(0)}(x) = -\not\partial\frac{1}{4\pi^2x^2} = \not x/(2\pi^2x^4)$.
The free propagator $S^{(0)}(x-y;m)$ for a massive fermion is the Green's function for $\not\partial+m$. Since $(\not\partial+m)(-\not\partial+m) = -\partial^2+m^2$, $S^{(0)}(x;m) = (-\not\partial+m)\Delta^{(0)}(x;m)$. Inserting the asymptotic form for $\Delta^{(0)}(x;m)$ gives $S^{(0)}(x;m) \sim (-\not\partial+m)\, e^{-mr}\sqrt{2m}/(4\pi r)^{3/2} = \sqrt2\,(\not n+1)\, e^{-mr}\,(m/4\pi r)^{3/2}$, where $r=|x|$ and $n_\mu \equiv x_\mu/r$, neglecting relative corrections suppressed by additional powers of $1/(mr)$.
The propagator $S(x,y;m)$ for a massive fermion in a background gauge field is the Green's

function for $\not{D}[A] + m$. Now $(\not{D}[A] + m)(-\not{D}[A] + m) = -D[A]^2 + m^2 - \frac{1}{2}\sigma^{\mu\nu} F_{\mu\nu}$, where $\gamma^\mu\gamma^\nu = g^{\mu\nu} + \sigma^{\mu\nu}$ with $\sigma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu]$, and $[D_\mu, D_\nu] = F_{\mu\nu}$ was used. In other words, because of the magnetic moment of fermions (which leads to a direct coupling between spin and magnetic fields), the fermion propagator in a background field is *not* just $(-\not{D}[A] + m)$ times the corresponding scalar propagator, rather it is $(-\not{D}[A] + m)$ times the Green's function for $-D[A]^2 + m^2 - \frac{1}{2}\sigma^{\mu\nu} F_{\mu\nu}$. Generalizing the previous path integral construction of the Green's function to this case leads to a path integral with the action $S[z] \equiv \int_0^T d\tau \left[\frac{1}{2}m(dz/d\tau)^2 + A(z(\tau)) \cdot (dz/d\tau) + \frac{1}{2}m - \frac{1}{4m}\sigma^{\mu\nu}F_{\mu\nu}\right]$. This shows that spin-dependent effects are suppressed by $1/m$. The leading large distance behavior of this path integral will still be proportional to $e^{-m|x-y|}$ times the parallel transporter $U(x, y)$, but now times the Lorentz transformation matrix $\Sigma(x, y) \equiv \exp\{\sigma^{\mu\nu}\bar{F}_{\mu\nu}\frac{|x-y|}{4m}\}$, where $\bar{F}_{\mu\nu}$ is the average of $F_{\mu\nu}(x)$ along the straight line path from x to y. Hence $S(x, y; m) \sim (-\not{D}[A] + m)\, e^{-m|x-y|}\, U(x, y)\, \Sigma(x, y)$.

3. $Q \equiv -\frac{1}{32\pi^2}\int d^4x\ \epsilon^{\mu\nu\alpha\beta}\,\mathrm{tr}\, F_{\mu\nu}F_{\alpha\beta}$, with $F_{\mu\nu} \equiv [D_\mu, D_\nu]$, $D_\mu \equiv \partial_\mu + A_\mu$, and $A_\mu \equiv A^a_\mu \frac{\tau_a}{2i}$ an anti-Hermitian SU(2) Lie-algebra-valued vector field.

   (a) Under an active Lorentz transformation $\Lambda_\mu{}^\nu$, the field strength tensor transforms like any rank-two tensor, $F_{\mu\nu}(x) \to \Lambda_\mu{}^{\mu'}\Lambda_\nu{}^{\nu'}F_{\mu'\nu'}(x')$, with $x'^\alpha \equiv \Lambda_\beta{}^\alpha x^\beta$. Therefore $\mathrm{tr}\ F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\,\mathrm{tr}\, F_{\mu\nu}(x)F_{\alpha\beta}(x) \to \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\,\Lambda_\mu{}^{\mu'}\Lambda_\nu{}^{\nu'}\Lambda_\alpha{}^{\alpha'}\Lambda_\beta{}^{\beta'}\,\mathrm{tr}\, F_{\mu'\nu'}(x')F_{\alpha'\beta'}(x')$. For any 4×4 matrix $\Lambda = \|\Lambda_\mu{}^\nu\|$, $\epsilon^{\mu\nu\alpha\beta}\,\Lambda_\mu{}^{\mu'}\Lambda_\nu{}^{\nu'}\Lambda_\alpha{}^{\alpha'}\Lambda_\beta{}^{\beta'} = \epsilon^{\mu'\nu'\alpha'\beta'}\det\Lambda$. (One way to verify this is to fully contract both sides with all possible products of eigenvectors of the matrix Λ.) Consequently, $\mathrm{tr}\ F_{\mu\nu}(x)\tilde{F}^{\mu\nu}(x) \to (\det\Lambda)\,\frac{1}{2}\epsilon^{\mu'\nu'\alpha'\beta'}\,\mathrm{tr}\, F_{\mu'\nu'}(x')F_{\alpha'\beta'}(x') = (\det\Lambda)\,\mathrm{tr}\ F_{\mu\nu}(x')\tilde{F}^{\mu\nu}(x')$. For a proper Lorentz transformation, $\det\Lambda = 1$, while for improper transformations, $\det\Lambda = -1$. So $\mathrm{tr}\ F_{\mu\nu}\tilde{F}^{\mu\nu}$ transforms like a scalar field under proper Lorentz transformations, but acquires an extra minus sign under improper transformations — and hence is a pseudoscalar field.

   (b) Under an infinitesimal variation of the gauge field, $A_\mu \to A_\mu + \delta A_\mu$ and $F_{\mu\nu} \to F_{\mu\nu} + \delta F_{\mu\nu}$, with $\delta F_{\mu\nu} = D^{\mathrm{adj}}_\mu\,\delta A_\nu - D^{\mathrm{adj}}_\nu\,\delta A_\mu$, where $D^{\mathrm{adj}}_\mu f \equiv [D_\mu, f] = \partial_\mu f + [A_\mu, f]$ is the adjoint-representation covariant derivative which is appropriate for any Lie-algebra-valued field f. Hence, $\delta Q = -\frac{\epsilon^{\mu\nu\alpha\beta}}{16\pi^2}\int d^4x\,\mathrm{tr}\,\delta F_{\mu\nu}\,F_{\alpha\beta} = \frac{\epsilon^{\mu\nu\alpha\beta}}{8\pi^2}\int d^4x\,\mathrm{tr}\,(D^{\mathrm{adj}}_\nu\,\delta A_\mu)\,F_{\alpha\beta} = -\frac{\epsilon^{\mu\nu\alpha\beta}}{8\pi^2}\int d^4x\,\mathrm{tr}\,\delta A_\mu\,D^{\mathrm{adj}}_\nu F_{\alpha\beta}$, where the covariant derivative was integrated by parts in the last step. This is valid, inside the trace, for the commutator part of the covariant derivative since $\mathrm{tr}\,([A_\nu, f]\,g) = -\mathrm{tr}\,(f\,[A_\nu, g])$. Finally, any gauge field satisfies the Bianchi identity $\epsilon^{\mu\nu\alpha\beta}\,D^{\mathrm{adj}}_\nu F_{\alpha\beta} = \epsilon^{\mu\nu\alpha\beta}\,[D_\nu, [D_\alpha, D_\beta]] = 0$, and hence $\delta Q = 0$. (The Bianchi identity follows from the fact that $\epsilon^{\mu\nu\alpha\beta}$ is symmetric under cyclic permutations of three indices, while double commutators satisfy the Jabobi identity, $[D_\nu, [D_\alpha, D_\beta]] + [D_\alpha, [D_\beta, D_\nu]] + [D_\beta, [D_\nu, D_\alpha]] = 0$.)
   The only subtlety in this argument is the integration-by-parts — this can produce a surface term which was neglected above, so that $\delta Q = \frac{\epsilon^{\mu\nu\alpha\beta}}{8\pi^2}\int d^3\Sigma_\nu\,\mathrm{tr}\,(\delta A_\mu\,F_{\alpha\beta})$, where the surface integral is over the sphere at (spacetime) infinity, or in other words the $R \to \infty$ limit of the integral over a 3-sphere of radius R, whose surface area grows as R^3. But if $F_{\mu\nu}(x)$ vanishes faster than $1/|x|^2$ and $\delta A_\mu(x)$ vanishes faster than $1/|x|$ as $x \to \infty$, then the integrand falls faster than $1/|x|^3$, so this $R \to \infty$ limit vanishes. So, under the stated conditions, any continuous variation of the gauge field leaves the value of the topological charge Q unchanged. Which means its value can only depend on topological properties of the gauge field — hence the name 'topological charge'.

   (c) If $-\frac{1}{16\pi^2}\,\mathrm{tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu} = \partial_\mu\, j^\mu$ for some current j^μ, then this current must be a pseudo-vector with dimension three constructed from the (dimension one) gauge field and derivatives. Possible terms are $\epsilon^{\mu\nu\alpha\beta}\,\mathrm{tr}\,(A_\nu\partial_\alpha A_\beta)$ and $\epsilon^{\mu\nu\alpha\beta}\,\mathrm{tr}\,(A_\nu A_\alpha A_\beta)$. (Any term with two derivatives would vanish upon contraction with the ϵ tensor.) The first term has the right structure so that its

divergence can generate the $(\partial A)(\partial A)$ part of $\mathrm{tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu}$, while the second term has right structure to generate the $(\partial A)AA$ pieces of $\mathrm{tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu}$. But what about parts of $\mathrm{tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu}$ which are quartic in the gauge field, namely $\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\,\mathrm{tr}\,[A_\mu, A_\nu][A_\alpha, A_\beta]$? Such quartic terms cannot be generated by the divergence of a current, and therefore $\mathrm{tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu}$ could not equal $\partial_\mu j^\mu$ for any current j^μ — unless these quartic terms are actually absent. To see if they do vanish, expand the commutators, consider any of the resulting terms such as $\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\,\mathrm{tr}\,(A_\mu\, A_\nu\, A_\alpha\, A_\beta)$, and note that the epsilon tensor changes sign under a cyclic permutation of all four indices, while the trace is symmetric. [*E.g.*, $\mathrm{tr}\,(A_\mu\, A_\nu\, A_\alpha\, A_\beta) = \mathrm{tr}\,(A_\nu\, A_\alpha\, A_\beta\, A_\mu)$.] Hence $\mathrm{tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu}$ does not, in fact, contain any terms quartic in the gauge field.

Let $j^\mu = \epsilon^{\mu\nu\alpha\beta}\,\{C_1\,\mathrm{tr}\,(A_\nu\partial_\alpha A_\beta) + C_2\,\mathrm{tr}\,(A_\nu A_\alpha A_\beta)\}$ and note that

$$\begin{aligned}\partial_\mu j^\mu &= \epsilon^{\mu\nu\alpha\beta}\,\{C_1\,\mathrm{tr}\,(\partial_\mu A_\nu)(\partial_\alpha A_\beta) + 3\,C_2\,\mathrm{tr}\,(\partial_\mu A_\nu)A_\alpha A_\beta\}\\ &= \epsilon^{\mu\nu\alpha\beta}\,\left\{\tfrac{1}{4}\,C_1\,\mathrm{tr}\,(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\alpha A_\beta - \partial_\beta A_\alpha) + \tfrac{3}{4}\,C_2\,\mathrm{tr}\,(\partial_\mu A_\nu - \partial_\nu A_\mu)[A_\alpha, A_\beta]\right\},\end{aligned}$$

while $\mathrm{tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\,\{\mathrm{tr}\,(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\alpha A_\beta - \partial_\beta A_\alpha) + 2\,\mathrm{tr}\,(\partial_\mu A_\nu - \partial_\nu A_\mu)[A_\alpha, A_\beta]\}$. So $-\frac{1}{16\pi^2}\,\mathrm{tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu}$ will equal $\partial_\mu j^\mu$ if $C_1 = -2/(16\pi^2)$ and $C_2 = -\frac{4}{3}/(16\pi^2)$. Hence

$$j^\mu = \frac{-1}{16\pi^2}\,\epsilon^{\mu\nu\alpha\beta}\left\{2\,\mathrm{tr}\,(A_\nu\partial_\alpha A_\beta) + \tfrac{4}{3}\,\mathrm{tr}\,(A_\nu A_\alpha A_\beta)\right\} = \frac{-1}{16\pi^2}\,\epsilon^{\mu\nu\alpha\beta}\,\mathrm{tr}\,(A_\nu F_{\alpha\beta} - \tfrac{2}{3}\,A_\nu A_\alpha A_\beta)\,.$$

(d) If $A_\mu(x) \to g(x)\partial_\mu g(x)^{-1}$ as $x \to \infty$, then this means that A_μ becomes a gauge transformation of zero as $x \to \infty$. And that implies that the field strength $F_{\mu\nu}$ vanishes as $x \to \infty$. Conversely, if the gauge field did not approach a pure gauge for large x, then the field strength wouldn't vanish as $x \to \infty$, contrary to the stated assumptions.

(e) Combine the results of the previous two parts to find:

$$\begin{aligned}Q &= \int d^4x\,\partial_\mu j^\mu(x) = \lim_{R\to\infty}\int d^3\Sigma_\mu\, j^\mu(x) = \frac{1}{24\pi^2}\int d^3\Sigma_\mu\,\epsilon^{\mu\nu\alpha\beta}\,\mathrm{tr}\,(A_\nu A_\alpha A_\beta)\\ &= \frac{1}{24\pi^2}\int d^3\Sigma_\mu\,\epsilon^{\mu\nu\alpha\beta}\,\mathrm{tr}\,\left\{[g\,\partial_\nu g^{-1}][g\,\partial_\alpha g^{-1}][g\,\partial_\beta g^{-1}]\right\},\end{aligned}$$

where the surface integral is over the sphere at spacetime infinity. Since $F_{\alpha\beta}$ falls faster than $1/x^2$ and A_ν will fall like $1/|x|$, the $\epsilon^{\mu\nu\alpha\beta}\,\mathrm{tr}\,(A_\nu F_{\alpha\beta})$ part of j^μ cannot contribute to the surface integral, and hence was dropped. [This asymptotic behavior for A_ν assumes that $g(x)$ approaches some element of the gauge group $g(\hat{x})$ depending only on direction as $x \to \infty$.]

(f) If $g(x) \to g_0$ as $x \to \infty$, then its angular derivatives will vanish and hence, from the above formula, so will the topological charge Q.

(g) Define $\bar{\tau}_\mu$ such that $\bar{\tau}_i$ is the usual i'th Pauli matrix, while $\bar{\tau}_0 \equiv -i$. And define τ_μ such that $\tau_0 = +i$ while τ_i are again the usual Pauli matrices. Note that $\tau_\mu = (\bar{\tau}_\mu)^\dagger$. And also observe that $(\bar{\tau}_\mu x^\mu)(\tau_\nu x^\nu) = (-ix^0 + \underset{\sim}{\tau}\cdot\underset{\sim}{x})(ix^0 + \underset{\sim}{\tau}\cdot\underset{\sim}{x}) = (x^0)^2 + (\underset{\sim}{x})^2 = x^2$. Hence, if $g(x) \to i\tau_\mu x^\mu/|x|$ at large x, then $g(x)^{-1} \to -i\bar{\tau}_\mu x^\mu/|x|$. And therefore, $g\,\partial_\alpha g^{-1} \to (\tau\cdot x\,\bar{\tau}_\alpha - x_\alpha)/x^2$ (with $\tau\cdot x \equiv \tau_\beta x^\beta$). The formula for topological charge thus gives $Q = \frac{1}{24\pi^2}\int d^3\Sigma_\mu\,\epsilon^{\mu\nu\alpha\beta}\,\mathrm{tr}\,(\tau\cdot x\,\bar{\tau}_\nu\,\tau\cdot x\,\bar{\tau}_\alpha\,\tau\cdot x\,\bar{\tau}_\beta)\,/x^6$. The x_α/x^2 part of $g\,\partial_\alpha g^{-1}$ doesn't contribute since the normal to the surface is purely radial and hence the epsilon tensor ensures that only angular components of $g\,\partial g^{-1}$ contribute to the integrand.

To simplify the trace, notice that $\tau_\alpha\bar{\tau}_\beta + \tau_\beta\,\bar{\tau}_\alpha = 2\,\delta_{\alpha\beta}$. Hence $\mathrm{tr}\,(\tau\cdot x\,\bar{\tau}_\nu\,\tau\cdot x\,\bar{\tau}_\alpha\,\tau\cdot x\,\bar{\tau}_\beta) = \mathrm{tr}\,[(-\tau_\nu\,\bar{\tau}\cdot x + 2\,x_\nu)\,\tau\cdot x\,\bar{\tau}_\alpha\,\tau\cdot x\,\bar{\tau}_\beta] = -x^2\,\mathrm{tr}\,(\tau_\nu\,\bar{\tau}_\alpha\,\tau\cdot x\,\bar{\tau}_\beta) + 2x_\nu\,\mathrm{tr}\,(\tau\cdot x\,\bar{\tau}_\alpha\,\tau\cdot x\,\bar{\tau}_\beta)$, and once again, the second term proportional to x_ν will vanish when contracted with $d^3\Sigma_\mu\epsilon^{\mu\nu\alpha\beta}$. So now $Q = -\frac{1}{24\pi^2}\int d^3\Sigma_\mu\,\epsilon^{\mu\nu\alpha\beta}\,\mathrm{tr}\,(\tau_\nu\,\bar{\tau}_\alpha\,\tau\cdot x\,\bar{\tau}_\beta)\,/x^4 = -\frac{1}{24\pi^2}\int d^3\Sigma_\mu\,\frac{x^\rho}{x^4}\,\epsilon^{\mu\nu\alpha\beta}\,\mathrm{tr}\,(\tau_\nu\,\bar{\tau}_\alpha\,\tau_\rho\,\bar{\tau}_\beta)$. At this

point, it is convenient to do the surface integral. By $O(4)$ rotational invariance, the integral $\int d^3\Sigma_\mu \, \frac{x^\rho}{x^4}$ must be proportional to δ^ρ_μ. Contracting the integral with δ^ρ_μ gives $\int d^3\Sigma_\mu \, \frac{x^\mu}{x^4} = \lim_{R\to\infty} \int d^3\Sigma \cdot \hat{x}/R^3 = 2\pi^2$ since the surface area of a three-sphere is $2\pi^2 \, R^3$. Hence the original integral $\int d^3\Sigma_\mu \, \frac{x^\rho}{x^4}$ equals $\frac{1}{2}\pi^2 \, \delta^\rho_\mu$, and this gives us $Q = -\frac{1}{48} \, \epsilon^{\mu\nu\alpha\beta} \, \mathrm{tr} \, (\tau_\nu \, \bar{\tau}_\alpha \, \tau_\mu \, \bar{\tau}_\beta)$. If $\mu\nu\alpha\beta$ equals 0123, then the epsilon tensor is $+1$ and the trace is $\mathrm{tr}\,(i\sigma_1\sigma_2\sigma_3) = \mathrm{tr}\,(-1) = -2$. If $\mu\nu\alpha\beta$ is any even permutation of 0123, the result is exactly the same, while for odd permutations the epsilon tensor is -1 while the trace is $+2$. Hence $Q = -\frac{1}{48} \, 4! \, (-2) = +1$.

(h) The space of finite action field configurations may be partitioned into distinct topological categories. We have shown that at least two such categories exist — those in which $Q = 0$ and $Q = +1$. If we had considered configurations with $g(x) \to -i\tau_\mu x^\mu/|x|$ as $x \to \infty$ (which is the inverse of $i\bar{\tau}_\mu x^\mu/|x|$) then we would have found a third category of configurations with $Q = -1$.

The gauge transformation $g(x)$ characterizing the asymptotic behavior of the gauge field gives a mapping from the sphere at infinity into the gauge group. The group $SU(2)$ is topologically a three-sphere, and hence the relevant topological classification is $\pi_3(S^3)$, the homotopy group of mappings of three-spheres into three-spheres. [To see that $SU(2)$ has the topology of S^3, note that any 2×2 unitary matrix with determinant one may be written as $i\tau_\mu n^\mu$ with n a unit length four-vector.] One may show that Q can take on arbitrary integer values and that $\pi_3(S^3) = \mathbb{Z}$, the set of integers. The existence of configurations for which Q is an arbitrary integer follows from what we have already shown, plus the fact that homotopy groups are truly groups, since one may compose two mappings to obtain a third and the topological classification must preserve this group structure.

(i) Since the topological charge Q is the spacetime integral of pseudoscalar field, it is invariant under proper Lorentz transformations, but changes sign under improper transformations such as parity or time-reversal. Therefore, if the term $i\theta Q$ is added to the Euclidean action then, for non-zero values of θ, parity and time-reversal will not be symmetries of the action. But because Q is quantized (*i.e.*, it only takes on integer values), at $\theta = \pm\pi$ the change in the action under an improper Lorentz transformation, for any field configuration, will be an integer multiple of $2\pi i$. Since the integrand of the functional integral is the exponential of the action, $e^{-S_E[A_\mu]}$, this means that the theory (as defined by this functional integral) is parity and time-reversal invariant at either $\theta = 0$ or $\theta = \pi$.

(j) Equations of motion are conditions for stationarity of the action under infinitesimal variations of the fields. We showed in part (b) that Q is unchanged under infinitesimal variations. Hence, adding $i\theta Q$ to the action has no effect on the classical equations of motion. Perturbation theory is equivalent to a saddle-point expansion of the functional integral about classical vacuum configurations. Since vacuum configurations are gauge-equivalent to $A_\mu = 0$, these minima of the action lie in the $Q = 0$ sector of the theory. The existence of topologically disjoint sectors of configuration space (in which Q is non-zero) has no effect whatsoever on the saddle-point expansion about vacuum configurations. Hence perturbation theory cannot depend on θ. But the complete functional integral includes the topologically non-trivial sectors, whose contributions will be weighted by the phase factor $e^{i\theta Q}$. Therefore, changing the value of θ will effect the value of the partition function, and hence will effect the ground state energy. Because Q only takes on integer values, the phase factor $e^{i\theta Q}$ is periodic in θ with period 2π. Hence, the theta dependence of the ground state energy must also be periodic with period 2π. Since the integrand of the functional integral is strictly positive at $\theta = 0$, introducing a non-zero θ, which gives a non-zero phase to contributions of topologically non-trivial configurations, can only decrease the total partition function. Since $E_{\text{g.s.}} \propto -\ln Z$, this means that the ground state energy, as a function of θ, must be minimized at $\theta = 0$.