1. Noether's Theorem. Consider a local field theory with a Lagrange density $\mathscr{L}$ which is a functional of some set of fields $\{\phi_a(x)\}$ and their derivatives. Assume for simplicity that $\mathscr{L}$ only depends on first derivatives of the fields. Let $\phi_a(x) \to \phi_a(x) + \epsilon\, \Delta\phi_a(x)$ be an infinitesimal transformation of the fields which leaves the action invariant, $\partial S/\partial\epsilon = 0$ (given appropriate boundary conditions on the fields).

   (a) Explain why the change in the Lagrange density under this transformation must be expressible as the divergence of some vector field (which is built from the fundamental fields), $\mathscr{L}(x) \to \mathscr{L}(x) + \epsilon\, \partial_\mu f^\mu(x)$.

   (b) Show that this invariance of the action implies that the current $j^\mu(x) \equiv \frac{\delta\mathscr{L}}{\delta[\partial_\mu\phi_a(x)]}\, \Delta\phi_a(x) - f^\mu(x)$ is conserved, $\partial_\mu j^\mu(x) = 0$, when the equations of motion for $\phi_a(x)$ are satisfied. This is Noether's theorem: every continuous symmetry in a local field theory is associated with a conserved current.

   (c) Define a "charge" Q as the spatial integral of the time component of the current, $Q \equiv \int d^3x\, j^0(x)$. Show that Q is constant in time (given reasonable boundary conditions at infinity) if the current is conserved.

   (d) In the quantized theory, show that the operator Q is the generator of the transformation, $i\,[Q, \phi_a(x)] = \Delta\phi_a(x)$, provided $\Delta\phi$ does not depend on time derivatives of the field.

   (e) In a theory of a complex scalar field with Lagrange density $|\partial\phi|^2 + V(|\phi|)$, What is the conserved current associated with the symmetry $\phi(x) \to e^{i\alpha}\phi(x)$?

   (f) Explain how Noether's theorem shows that space-time translation invariance implies the existence of a conserved energy-momentum tensor satisfying $\partial_\mu T^{\mu\nu}(x) = 0$. What is $T^{\mu\nu}$ in the above scalar field theory?

   (g) What is $T^{\mu\nu}$ in a theory of fermions with Lagrange density $\bar\psi(\not\partial + m)\psi + \frac{1}{2}\lambda(\bar\psi\psi)^2$?

   (h) If there exists some other tensor $\Delta T^{\mu\nu}$ which is independently conserved, $\partial_\mu \Delta T^{\mu\nu}(x) = 0$, then one can always redefine the stress-energy tensor, $T^{\mu\nu}_{\text{improved}} = T^{\mu\nu} + \Delta T^{\mu\nu}$. In the above examples of scalar and spinor theories, show that one can define a stress-energy tensor which is symmetric, $T^{\mu\nu} = T^{\nu\mu}$.

   (i) What conserved currents are associated with Lorentz transformations?

2. "Twisted" partition functions. Consider any quantum theory in which there is some symmetry transformation $\hat T$ (*i.e.* a unitary operator commuting with the Hamiltonian), and assume that all integer powers $(\hat T)^n$, $n = \pm1, \pm2, \pm3\cdots$ are distinct transformations, so that the group of transformations generated by $\hat T$ is equivalent to the additive group of integers $\mathbb{Z}$.

   (a) What is a physical example of a theory with this structure?

   (b) Explain why this symmetry group implies that every eigenstate may be labeled by some angle θ (in addition to the energy).

   (c) Explain why $\hat P(\theta) \equiv \sum_n e^{-in\theta}\, \hat T^n$ is a projection operator onto states satisfying $\hat T|\psi\rangle = e^{i\theta}|\psi\rangle$. In other words, $P(\theta)$ projects onto the indicated eigenspace of $\hat T$. What is $\hat P(\theta)\hat P(\theta')$?

(d) Define a generalization of the usual partition function which includes a symmetry transformation inside the trace, $Z_n \equiv \mathrm{Tr}\, \hat{T}^n e^{-\beta \hat{H}}$. Alternatively, define a different generalization of the usual partition function which includes only the contributions of states in a specific eigenspace of $\hat{T}$, $Z(\theta) \equiv \mathrm{Tr}\, \hat{P}(\theta) e^{-\beta \hat{H}}$. What is the relation between Z_n and $Z(\theta)$?

(e) Consider one-dimensional quantum mechanics of a particle in a potential $V(x)$ which is periodic with period a, $V(x+a) = V(x)$, so that a spatial translation through distance a is a symmetry of the theory. What is the path integral representation of Z_n? What path(s), contributing to this path integral, have the minimum Euclidean action? (Describe them qualitatively, and characterize them quantitatively in terms of the differential equation they satisfy together with suitable boundary conditions.) Let $S_n(\beta)$ denote the minimal action of paths which contribute to Z_n. Without knowing the detailed form of the potential $V(x)$ (other than its periodicity, and boundedness from below), what can you deduce about the dependence on $S_n(\beta)$ on n and β?

(f) Introduce an adjustable coupling constant g by multiplying the path-integral action by an overall factor of $1/g^2$. What does knowledge of $S_1(\beta)$ allow you to deduce about the dependence of $Z(\theta)$ on θ, when $g^2 \ll 1$? In the limit of large β, what does this imply about the θ dependence of energy levels of the Hamiltonian?