1. Consider an $SU(N)$ gauge theory containing N different scalar fields, each transforming in the fundamental representation. Assemble these scalar fields into a single $N \times N$ matrix $\Phi = \|\Phi_{ai}\|$, where a is a gauge group index and i is a flavor index, each running from 1 to N. Let

$$-\mathcal{L} = -\frac{1}{2g^2}\,\mathrm{tr}\,F_{\mu\nu}^2 + \mathrm{tr}\,[(D_\mu\Phi)^\dagger(D^\mu\Phi)] + m^2\,\mathrm{tr}\,\Phi^\dagger\Phi + \tfrac{1}{2N}\,\lambda_1\,(\mathrm{tr}\,\Phi^\dagger\Phi)^2 + \tfrac{1}{2}\,\lambda_2\,\mathrm{tr}\,[(\Phi^\dagger\Phi)^2]\,.$$

   (a) What are the symmetries (local and global) of this theory?

   (b) Assuming that $m^2 < 0$ while λ_1 and λ_2 are positive, show that the (Euclidean) action has a global minima, or vacuum configurations, where $\Phi(x) = v\,u(x)\,e^{i\chi(x)}$ and the gauge field has the pure-gauge form $A_\mu(x) = u(x)\partial_\mu u(x)^\dagger$, for some magnitude $v > 0$ and arbitrary $u(x) \in SU(N)$ and phase $\chi(x)$. What is v?

   (c) Under what subgroup of the full symmetry group is some given vacuum field configuration left invariant?

   (d) To expand the action about any of these vacuum configurations, let $\Phi(x) = u(x)\,e^{i\chi(x)}\,h(x)$ and $A_\mu(x) = u(x)(\partial_\mu + a_\mu(x))u(x)^\dagger$, where $u(x) \in SU(N)$, the phase χ is (of course) real, and $h(x)$ is a positive semi-definite Hermitian matrix. Explain why this decomposition is unique provided $\Phi(x)$ is invertible. (So, in particular, this decomposition is well behaved in the neighborhood of any vacuum configuration.)

   (e) Let $h(x) = v\,1_N + \delta h(x)$, where 1_N denotes the $N \times N$ identity matrix. Rewrite each term of the action in terms of the fields a_μ, χ, u and δh of this decomposition. Expand the resulting action in powers of δh, simplifying your expressions as much as possible.

   (f) What is the tree-level spectrum of this theory? I.e., what are the masses and spins of physical particles? How many massless Goldstone bosons are present?

   (g) What are the interaction terms involving Goldstone bosons? What physical scattering processes will these induce? How will the cross-section for each of these processes depend on the energy of the Goldstone boson(s)?

   (h) Now suppose that $m^2 > 0$. In this regime, what is the tree-level spectrum of the theory? How many massless Goldstone bosons are present? Must there be a phase transition (i.e., some point of non-analyticity in physical quantities) as m^2 changes from large negative to large positive values?

   (i) Finally, suppose the additional term $\delta\mathcal{L} = \epsilon\,(\det\Phi + \det\Phi^\dagger)$ is added to the theory, with ϵ small and negative. How does this change the symmetries of the theory? How will this perturbation affect your answers to parts (f) and (h)?

2. Consider an $SU(N)$ gauge theory with a single adjoint represntation Hermitian scalar field Φ, and Lagrange density

$$-\mathcal{L} = -\frac{1}{2g^2}\,\mathrm{tr}\,F_{\mu\nu}^2 + \tfrac{1}{2}\,\mathrm{tr}\,(D_\mu\Phi)^2 + \tfrac{1}{2}\,m^2\,\mathrm{tr}\,\Phi^2 + \tfrac{1}{4N}\,\lambda_1\,(\mathrm{tr}\,\Phi^2)^2 + \tfrac{1}{4}\,\lambda_2\,\mathrm{tr}\,\Phi^4\,.$$

   (a) What are the symmetries (local and global) of this theory?

(b) Assuming that $m^2 < 0$ while λ_1 and λ_2 are positive, what field configurations minimize the action? What subgroup of the full symmetry group leaves a vacuum field configuration invariant?

(c) Expand the action about any of these vacuum configurations. Simplify your results.

(d) What is the tree-level spectrum of the theory? What are the low-energy degrees of freedom? How many massless Goldstone bosons are present?

(e) What changes if $m^2 > 0$? Must there be a phase transition as m^2 changes from large negative to large positive values?