1. (a) If a field configuration $\phi_a(x)$ is a stationary point of the action (*i.e.*, a solution to the equations of motion), then the transformed field $\phi_a(x) \to \phi_a(x) + \epsilon\,\Delta\phi_a(x)$, must also be a stationary point if this transformation (for any configuration) leaves the action invariant. So if $\mathscr{L} \to \mathscr{L}' = \mathscr{L} + \epsilon\,\Delta\mathscr{L}$ under this transformation, then $\mathscr{L}$ and $\mathscr{L}'$ must generate the same equations of motion. This will be the case if $\Delta\mathscr{L}$ is a total derivative (*i.e.*, a divergence of a vector field), $\Delta\mathscr{L}(x) = \partial_\mu f^\mu(x)$, so that its spacetime integral just gives a surface term (which will vanish given appropriate boundary conditions on the fields).

   (b) If $j^\mu(x) \equiv \frac{\delta\mathscr{L}}{\delta[\partial_\mu\phi_a(x)]}\,\Delta\phi_a(x) - f^\mu(x)$ then the divergence of the current j^μ is

$$\begin{aligned}
\partial_\mu j^\mu(x) &= \partial_\mu\Big(\frac{\delta\mathscr{L}}{\delta[\partial_\mu\phi_a(x)]}\,\Delta\phi_a(x)\Big) - \partial_\mu f^\mu(x) \\
&= \Big[\partial_\mu\Big(\frac{\delta\mathscr{L}}{\delta[\partial_\mu\phi_a(x)]}\Big)\Big]\Delta\phi_a(x) + \frac{\partial\mathscr{L}}{\partial[\partial_\mu\phi_a(x)]}\,\partial_\mu\Delta\phi_a(x) - \partial_\mu f^\mu(x) \\
&= \frac{\delta\mathscr{L}}{\delta\phi_a(x)}\,\Delta\phi_a(x) + \frac{\delta\mathscr{L}}{\delta[\partial_\mu\phi_a(x)]}\,\partial_\mu\Delta\phi_a(x) - \partial_\mu f^\mu(x) \\
&= \Delta\mathscr{L} - \partial_\mu f^\mu(x) \\
&= 0\,,
\end{aligned}$$

   where the middle step assumes that $\phi_a(x)$ satisfies the Lagrange equation of motion, namely $\partial_\mu\frac{\delta\mathscr{L}}{\delta[\partial_\mu\phi_a(x)]} - \frac{\delta\mathscr{L}}{\delta\phi_a(x)} = 0$.

   (c) As with any conserved current, the time derivative of the charge $Q \equiv \int d^3x\; j^0(x)$ is $\frac{dQ}{dt} = \int d^3x\,\partial_0 j^0(x) = \int d^3x\;[\partial_\mu j^\mu(x) - \underset{\sim}{\nabla}\cdot\underset{\sim}{j}(x)] = \int d^3x\,\partial_\mu j^\mu(x) - \int_{S_\infty} d^2\underset{\sim}{\Sigma}\cdot\underset{\sim}{j}(x)$, where the surface integral is over a sphere at spatial infinity. Therefore $dQ/dt = 0$ if the current is conserved, $\partial_\mu j^\mu = 0$ and the spatial current (density) $\underset{\sim}{j}(x)$, which is the flux of the charge density $j^0(x)$, vanishes at spatial infinity.

   (d) In the classical field theory, the conjugate momentum $\Pi_a(\underset{\sim}{x}) = \delta\mathscr{L}/\delta[\partial_0\phi_a(\underset{\sim}{x})]$, and so the conserved charge $Q = \int d^3x\; j^0(\underset{\sim}{x}) = \int d^3x\;[\Pi_a(\underset{\sim}{x})\Delta\phi_a(\underset{\sim}{x}) - f^0(\underset{\sim}{x})]$. Converting to a Hamiltonian treatment, where $\phi_a(\underset{\sim}{x})$ and $\Pi_a(\underset{\sim}{x})$ (at equal times) are regarded as independent, the Poisson bracket $\{Q, \phi_b(\underset{\sim}{y})\}_{\rm PB} = \int d^3x\;\{\Pi_a(\underset{\sim}{x}), \phi_b(\underset{\sim}{y})\}_{\rm PB}\,\Delta\phi_a(\underset{\sim}{x}) = \int d^3x\;\;\delta_{ab}\delta^3(\underset{\sim}{x}-\underset{\sim}{y})\;\rangle\Delta\phi_a(\underset{\sim}{x}) = \Delta\phi_b(\underset{\sim}{y})$, provided $f^0(\underset{\sim}{x})$ and $\Delta\phi_a(\underset{\sim}{x})$ do not depend on time derivatives of the field (or equivalently do not depend on the conjugate momentum $\Pi_a(\underset{\sim}{x})$), so that their Poisson brackets with the field vanish. Assuming that this is the case, this shows that the charge Q is the generator of the symmetry transformation $\phi_a(x) \to \phi_a(x) + \epsilon\Delta\phi_a(x)$.

   Quantizing the theory makes $\phi_a(x)$ and $\Pi_a(x)$ quantum operators and replaces Poisson brackets with (i times) commutators, so that $i[\hat{Q}, \hat{\phi}_a(\underset{\sim}{x})] = \Delta\hat{\phi}_a(\underset{\sim}{x})$ — with the same interpretation.

   (e) The Lagrange density $\mathscr{L} = |\partial\phi|^2 + V(|\phi|)$ is invariant under the phase rotation symmetry $\phi \to e^{i\alpha}\phi$. So the associated conserved current is $j^\mu(x) = \frac{\delta\mathscr{L}}{\delta[\partial_\mu\phi(x)]}\,\Delta\phi(x) + \frac{\delta\mathscr{L}}{\delta[\partial_\mu\phi^*(x)]}\,\Delta\phi^*(x) = (\partial^\mu\phi^*(x))(i\phi(x)) + (\partial^\mu\phi(x))(-i\phi^*(x)) = -i\,[\phi(x)^*(\partial^\mu\phi(x)) - (\partial^\mu\phi^*(x))\phi(x)]$. (The complex field ϕ could have been separated into independent real and imaginary parts. This is equivalent to treating ϕ and ϕ^* as independent fields, as done here.)

(f) An arbitrary translation may be regarded as a combination of translations along each of the coordinate directions. For a spacetime translation in the ν-direction, the (infinitesimal) transformation of any field $\phi(x)$ is $\phi(x) \to \phi(x + \epsilon\,\hat{n}_\nu) = \phi(x) + \epsilon\,\partial_\nu \phi(x)$, where $\hat{n}_\nu$ is a unit vector in the direction ν. For any translation invariant theory, this will be an invariance of the action. Therefore, as shown above, there will be an associated conserved current associated with translations along each coordinate direction. These d different currents [in d-dimensional spacetime] may be viewed as the columns of a single rank two tensor $T^{\mu\nu}$. The time components of these currents, integrated over space, are the conserved charges which, for translations, are just the components of the total momentum, $P_\nu = \int d^3x\, T^{0\nu}(x)$. [Note that this is completely independent of Lorentz invariance, it only depends on translation invariance.]

For the above scalar field theory, if $\phi(x) \to \phi(x) + \epsilon\, n^\nu \partial_\nu \phi(x)$, for some unit vector n^ν, then $\mathscr{L}(x) \to \mathscr{L}(x) + \epsilon\, n^\nu \partial_\nu \mathscr{L}(x) = \mathscr{L}(x) + \epsilon\, \partial_\mu f^\mu(x)$ with $f^\mu(x) = n^\mu \, \mathscr{L}(x)$. Noether's theorem then gives the associated conserved current

$$
\begin{aligned}
j^\mu &= \frac{\delta \mathscr{L}}{\delta[\partial_\mu \phi(x)]}\; n^\nu \partial_\nu \phi(x) + \frac{\delta \mathscr{L}}{\delta[\partial_\mu \phi^*(x)]}\; n^\nu \partial_\nu \phi^*(x) - n^\mu\, \mathscr{L}(x) \\
&= \partial^\mu \phi^*(x)\, n^\nu \partial_\nu \phi(x) + \partial^\mu \phi(x)\, n^\nu \partial_\nu \phi^*(x) - n^\mu\, \mathscr{L}(x) \\
&= \{[\partial^\mu \phi^*(x)][\partial^\nu \phi(x)] + [\partial^\mu \phi(x)][\partial^\nu \phi^*(x)] - g^{\mu\nu}\, \mathscr{L}(x)\}\, n_\nu \\
&\equiv T^{\mu\nu}(x) n_\nu
\end{aligned}
$$

Hence $T^{\mu\nu} = (\partial^\mu \phi^*)(\partial^\nu \phi) + (\partial^\nu \phi^*)(\partial^\mu \phi) - g^{\mu\nu}\, \mathscr{L}$. It may be instructive to verify the conservation of $T^{\mu\nu}$ directly. We have $\partial_\mu T^{\mu\nu} = (\partial^2 \phi^*)(\partial^\nu \phi) + (\partial^\nu \phi^*)(\partial^2 \phi) + \partial^\nu [(\partial^\mu \phi^*)(\partial_\mu \phi) - \mathscr{L}]$. The last term is $-\partial^\nu V(|\phi|) = -\frac{V'(|\phi|)}{2|\phi|}\,[\phi^* \partial^\nu \phi + \phi\, \partial^\nu \phi^*]$, so that $\partial_\mu T^{\mu\nu} = \left[\partial^2 \phi^* - \frac{V'(|\phi|)}{2|\phi|}\,\phi^*\right](\partial^\nu \phi) + (\partial^\nu \phi^*)\left[\partial^2 \phi - \frac{V'(|\phi|)}{2|\phi|}\,\phi\right]$. The equation of motion is $\left[\partial^2 - \frac{V'(|\phi|)}{2|\phi|}\right]\phi = 0$, and hence $\partial_\mu T^{\mu\nu}$ vanishes, as required, when ϕ (and ϕ^*) satisfy their equations of motion.

(g) For a fermionic theory with Lagrange density $\bar{\psi}(\partial\!\!\!/ + m)\psi + \frac{1}{2}\lambda(\bar{\psi}\psi)^2$, the same Noether's theorem construction of the stress-energy tensor gives

$$
\begin{aligned}
T^{\mu\nu} &= \frac{\delta \mathscr{L}}{\delta[\partial_\mu \psi(x)]}\; \partial^\nu \psi(x) + \frac{\delta \mathscr{L}}{\delta[\partial_\mu \bar{\psi}(x)]}\; \partial^\nu \bar{\psi}(x) - g^{\mu\nu}\, \mathscr{L}(x) \\
&= \bar{\psi}\, \gamma^\mu\, \partial^\nu \psi(x) - g^{\mu\nu}\, \mathscr{L}(x) \\
&= \bar{\psi}\, \gamma^\mu\, \partial^\nu \psi(x) - g^{\mu\nu}\left[\bar{\psi}(\partial\!\!\!/ + m)\psi + \tfrac{1}{2}\lambda(\bar{\psi}\psi)^2\right].
\end{aligned}
$$

Note that for this Lagrange density (with the derivative acting to the right on ψ), $\frac{\delta \mathscr{L}}{\delta[\partial_\mu \bar{\psi}(x)]} = 0$. Since the equation of motion for ψ is $(\partial\!\!\!/ + m + 2\bar{\psi}\psi)\psi = 0$, this stress-energy tensor may be simplified to $T^{\mu\nu} = \bar{\psi}\, \gamma^\mu\, \partial^\nu \psi(x) + \frac{1}{2} g^{\mu\nu}\, \lambda(\bar{\psi}\psi)^2$.

Alternatively if we had integrated by parts in the action and started with the different (but equivalent) form $\mathscr{L} = \bar{\psi}(-\overleftarrow{\partial\!\!\!/} + m)\psi + \frac{1}{2}\lambda(\bar{\psi}\psi)^2$, where $f\, \overleftarrow{\partial}_\mu g \equiv (\partial_\mu f) g$, then Noether's theorem would have given a different result, $T^{\mu\nu} = -\bar{\psi}\, \gamma^\mu\, \overleftarrow{\partial}^\nu \psi(x) - g^{\mu\nu}\left[\bar{\psi}(-\overleftarrow{\partial\!\!\!/} + m)\psi + \frac{1}{2}\lambda(\bar{\psi}\psi)^2\right]$. If one averages the two (equally valid) forms, so that

$\mathscr{L} = \bar\psi(\overleftrightarrow{\not\partial} + m)\psi + \frac{1}{2}\lambda(\bar\psi\psi)^2$, with $f \overleftrightarrow{\partial}_\mu g \equiv \frac{1}{2} f(\partial_\mu g) - \frac{1}{2}(\partial_\mu f) g$, then Noether's theorem gives a Hermitian result, $T^{\mu\nu} = \bar\psi\,\gamma^\mu\,\overleftrightarrow{\partial}^\nu\psi(x) - g^{\mu\nu}\big[\bar\psi(\overleftrightarrow{\not\partial}+m)\psi + \frac{1}{2}\lambda(\bar\psi\psi)^2\big]$. Verifying, by direct calculation, that $\partial_\mu T^\mu = 0$ if ψ and $\bar\psi$ satisfy their equations of motion is a good exercise.

(h) For the scalar theory, Noether's theorem directly yields a symmetric stress-energy tensor, so no improvement is needed. For the fermionic theory, the above stress-energy tensor (in any of the forms given) is not symmetric. One may, of course, decompose the result of Noether's theorem into a sum of symmetric and antisymmetric parts. Using the Hermitian form, we have $T^{\mu\nu} = T^{\mu\nu}_{\rm sym} + T^{\mu\nu}_{\rm asym}$ with $T^{\mu\nu}_{\rm sym} = \frac{1}{2}\bar\psi\,\gamma^\mu\,\overleftrightarrow{\partial}^\nu\psi(x) + \frac{1}{2}\bar\psi\,\gamma^\nu\,\overleftrightarrow{\partial}^\mu\psi(x) - g^{\mu\nu}\big[\bar\psi(\overleftrightarrow{\not\partial}+m)\psi + \frac{1}{2}\lambda(\bar\psi\psi)^2\big]$ and $T^{\mu\nu}_{\rm asym} = \frac{1}{2}\bar\psi\,\gamma^\mu\,\overleftrightarrow{\partial}^\nu\psi(x) - \frac{1}{2}\bar\psi\,\gamma^\nu\,\overleftrightarrow{\partial}^\mu\psi(x)$. One cannot just drop $T^{\mu\nu}_{\rm asym}$ — unless it can be shown that its divergence vanishes (when the equations of motion are satisfied). To see if this is the case, one can write out $\partial_\mu T^{\mu\nu}_{\rm asym}$ and try to manipulate the result into a form where one recognizes equation of motion terms (as was done above in part (e) for the scalar theory). It is equivalent, and perhaps easier, to work backwards. Suppose $A^{\mu\nu\lambda}$ is any third-rank tensor, which is antisymmetric under interchange of μ and λ, so $A^{\mu\nu\lambda} = -A^{\lambda\nu\mu}$. Then $\Delta T^{\mu\nu} \equiv \partial_\lambda A^{\mu\nu\lambda}$ is a rank-two tensor which is automatically divergenceless, $\partial_\mu(\Delta T^{\mu\nu}) = 0$. If there is some choice of $A^{\mu\nu\lambda}$ for which $\Delta T^{\mu\nu} = \partial_\lambda A^{\mu\nu\lambda}$ will cancel $T^{\mu\nu}_{\rm asym}$, then $A^{\mu\nu\lambda}$ will have to be a rank-three bilinear involving a product of three gamma matrices surrounded by $\bar\psi$ and ψ, but no derivatives. There are only a few possibilities with the required antisymmetry under $\mu \leftrightarrow \lambda$, and only one which is not reducible using the gamma matrix algebra, namely $A^{\mu\nu\lambda} = \frac{1}{4}\bar\psi\{\gamma^\nu, [\gamma^\mu, \gamma^\lambda]\}\psi$. Take the divergence (on λ), use the gamma matrix algebra to move the resulting $\not\partial$ to the right or left (depending on whether the derivative is acting on ψ or $\bar\psi$), and then use the equations of motion $(\not\partial + \tilde m)\psi = 0$ and $\bar\psi(-\overleftarrow{\not\partial} + \tilde m) = 0$, with $\tilde m \equiv m + 2\lambda\bar\psi\psi$. This yields $\partial_\lambda A^{\mu\nu\lambda} = 2\bar\psi(\gamma^\mu\overleftrightarrow{\partial}^\nu - \gamma^\nu\overleftrightarrow{\partial}^\mu)\psi = 4\,T^{\mu\nu}_{\rm asym}$. So subtracting $\frac{1}{4}\partial_\lambda A^{\mu\nu\lambda}$ from the (Hermitian) stress-energy tensor produced by Noether's theorem gives a symmetric and conserved stress-energy tensor, equal to the above $T^{\mu\nu}_{\rm sym}$.

(i) For an infinitesimal Lorentz transformation of a scalar field, $\delta\phi(x) = \frac{1}{2}\omega_{\alpha\beta}(x^\alpha\partial^\beta - x^\beta\partial^\alpha)\phi(x)$, where $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$ the infinitesimal parameters specifying the transformation. Under this transformation, the Lagrange density also (necessarily) transforms as a scalar field, so that $\delta\mathscr{L} = \frac{1}{2}\omega_{\alpha\beta}(x^\alpha\partial^\beta - x^\beta\partial^\alpha)\mathscr{L}$. This may be rewritten as $\delta\mathscr{L} = \partial^\beta(\omega_{\alpha\beta}\,x^\alpha\mathscr{L}) = \partial_\mu\,f^\mu$ with $f^\mu = \omega^{\alpha\mu}\,x_\alpha\mathscr{L}$. Applying the Noether procedure to this transformation, for the above scalar theory, produces the current $j^\mu = \frac{1}{2}\omega_{\alpha\beta}\,\mathscr{J}^{\,\mu,\alpha\beta}$ with $\mathscr{J}^{\,\mu,\alpha\beta} = (\partial^\mu\phi^*)(x^\alpha\partial^\beta - x^\beta\partial^\alpha)\phi + [(x^\alpha\partial^\beta - x^\beta\partial^\alpha)\phi^*]\partial^\mu\phi - (x^\alpha g^{\mu\beta} - x^\beta g^{\mu\alpha})\,\mathscr{L}$. This may be written more compactly in terms of the stress-energy tensor of the scalar theory, $\mathscr{J}^{\,\mu,\alpha\beta} = x^\alpha\,T^{\mu\beta} - x^\beta\,T^{\mu\alpha}$.

The change in a spinor field, under an infinitesimal Lorentz transformation, is $\delta\psi(x) = \frac{1}{2}\omega_{\alpha\beta}[x^\alpha\partial^\beta - x^\beta\partial^\alpha - \frac{i}{2}\sigma^{\alpha\beta}]\psi(x)$, where $\sigma^{\alpha\beta} \equiv \frac{i}{2}[\gamma^\alpha, \gamma^\beta]$. Applying the Noether procedure to this transformation, for the above spinor theory gives $j^\mu = \frac{1}{2}\omega_{\alpha\beta}\,\mathscr{J}^{\,\mu,\alpha\beta}$ with $\mathscr{J}^{\,\mu,\alpha\beta} = \bar\psi\gamma^\mu(x^\alpha\partial^\beta - x^\beta\partial^\alpha - \frac{i}{2}\sigma^{\alpha\beta})\psi - (x^\alpha g^{\mu\beta} - x^\beta g^{\mu\alpha})\,\mathscr{L}$.

Alternatively, if one starts from the symmetric stress-energy tensor, then one can always build a rank-3 angular momentum current via $\bar{\mathscr{J}}^{\,\mu,\alpha\beta} \equiv x^\alpha\,T^{\mu\beta}_{\rm sym} - x^\beta\,T^{\mu\alpha}_{\rm sym}$. This is conserved since $\partial_\mu\bar{\mathscr{J}}^{\,\mu,\alpha\beta} = T^{\alpha\beta}_{\rm sym} - T^{\beta\alpha}_{\rm sym} = 0$. For the spinor theory, the Noether procedure angular momentum current $\mathscr{J}^{\mu,\alpha\beta}$ differs from $\bar{\mathscr{J}}^{\mu,\alpha\beta}$ (just as the Noether $T^{\alpha\beta}$ differs from $T^{\alpha\beta}_{\text{sym}}$), but one can show that both forms of the angular momentum current lead to the same Lorentz generators $M^{\alpha\beta} \equiv \int d^3x \, \mathscr{J}^{0,\alpha\beta}$.

2. "Twisted" partition functions.

(a) A particle moving in a potential periodic with some lattice spacing a is an example of a quantum theory with a discrete translation symmetry. The translation operator $\hat{T} = \exp(ia\hat{p}/\hbar)$ performs a shift by one lattice spacing, $\hat{T}\,\hat{x}\,\hat{T}^\dagger = \hat{x} + a$, and will be a symmetry of the theory if all terms in the Hamiltonian are invariant under this shift.

(b) Since $[\hat{T}, \hat{H}] = 0$, we know that these operators are simultaneously diagonalizable. Since $\hat{T}$ is unitary, its eigenvalues must be pure phases. If some state $|\psi\rangle$ is a simultaneous eigenstate of $\hat{H}$ and $\hat{T}$, this implies that $\hat{H}|\psi\rangle = E|\psi\rangle$ for some (real) energy E, an $\hat{T}|\psi\rangle = e^{i\theta}\,|\psi\rangle$ for some (real) phase θ. In other words, eigenstates can be labeled by both their energy and their phase under the symmetry $\hat{T}$, $|\psi\rangle = |\psi_{E,\theta}\rangle$.

(c) Let $\hat{P}(\theta) \equiv \sum_n e^{-in\theta}\hat{T}^n$ act on some arbitrary initial state $|\Psi\rangle$, so $|\Psi_\theta\rangle \equiv \hat{P}(\theta)|\Psi\rangle$. Applying $\hat{T}$ to this state,

$$\hat{T}\,|\Psi_\theta\rangle = \hat{T}\sum_n e^{-in\theta}\hat{T}^n|\Psi\rangle = e^{i\theta}\sum_n e^{-i(n+1)\theta}\hat{T}^{n+1}|\Psi\rangle = e^{i\theta}\sum_m e^{-im\theta}\hat{T}^m|\Psi\rangle = e^{i\theta}|\Psi_\theta\rangle\,,$$

shows that $|\Psi_\theta\rangle$ is an eigenstate of $\hat{T}$ with eigenvalue $e^{i\theta}$. In other words, $\hat{P}(\theta)$ projects the complete Hilbert space onto a eigenspace of $\hat{T}$ with the specified eigenvalue.

An alternative way to see this is to consider any state $|\psi_{\theta'}\rangle$ which is an eigenstate of $\hat{T}$ with eigenvalue $e^{i\theta'}$. Applying $\hat{P}(\theta)$ to this state gives,

$$\hat{P}(\theta)|\psi_{\theta'}\rangle = \sum_n e^{-in\theta}\hat{T}^n|\psi_{\theta'}\rangle = \sum_n e^{-in(\theta-\theta')}|\psi_{\theta'}\rangle = 2\pi\delta(\theta-\theta')\,|\psi_{\theta'}\rangle\,,$$

where the basic Fourier series completeness relation $\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{-in(\theta-\theta')} = \delta(\theta-\theta')$ was used. So $\hat{P}(\theta)$ annihilates any state $\psi_{\theta'}$ if $\theta \neq \theta'$. Similarly,

$$\hat{P}(\theta)\hat{P}(\theta') = \sum_{n,m} e^{-in\theta}\hat{T}^n e^{-im\theta'}\hat{T}^m = \sum_m e^{-im(\theta'-\theta)}\sum_p e^{-ip\theta}\hat{T}^p = 2\pi\delta(\theta-\theta')\,\hat{P}(\theta)\,,$$

where in the second form $p = m + n$ was introduced, with the sum can be rewritten to be over p and m. The appearance of Dirac delta functions just reflects the fact that eigenspaces of $\hat{T}$ are labeled by a continuous parameter. Note the completeness relation, $\int \frac{d\theta}{2\pi}\,\hat{P}(\theta) = \sum_n \int \frac{d\theta}{2\pi}\,e^{-in\theta}\hat{T}^n = \sum_n \delta_{n,0}\,\hat{T}^n = 1$, showing that integrating the projections $\hat{P}(\theta)$ over all θ produces the identity operator.

(d) By explicit substitution,

$$Z(\theta) \equiv \mathrm{Tr}\,\hat{P}(\theta)\,e^{-\beta\hat{H}} = \sum_n e^{-in\theta}\,\mathrm{Tr}(\hat{T}^n e^{-\beta\hat{H}}) = \sum_n e^{-in\theta}\,Z_n,$$

so the Z_n are the Fourier coefficients of $Z(\theta)$.

(e) The path integral representation of Z_n is

$$Z_n = \mathrm{Tr}\,(\hat{T}^n e^{-\beta\hat{H}}) = \int dx_0\,\langle x_0|\hat{T}^n e^{-\beta\hat{H}}|x_0\rangle = \int dx_0\,\langle x_0{-}na|e^{-\beta\hat{H}}|x_0\rangle$$
$$= \int dx_0 \int_{x(0)=x_0}^{x(\beta)=x_0} \mathcal{D}x(\tau)\, e^{-S_E[x(\tau)]} = \int_{x(\beta)=x(0)-na} \mathcal{D}x(\tau)\, e^{-S_E[x(\tau)]}\,,$$

where $S_E[x(\tau)] \equiv \int_0^\beta d\tau\, \frac{1}{2m}(\frac{d}{d\tau}x(\tau))^2 + V(x(\tau))$ is the Euclidean action of a point particle in potential $V(x)$. For simplicity, henceforth, we set $m = 1$. The path integral representation of Z_n differs from the usual path integral for a partition function in that the relevant paths are not periodic, but rather are periodic up to a shift by $\Delta x \equiv na$.

Per our assumption about the potential, $V(x)$ has an infinite set of degenerate minima. Let $x_n \equiv x_0 + na$, for any $n \in \mathbb{Z}$, denote locations of these minima, with $V_0 \equiv V(x_n)$ denoting the minimum value of the potential. One can always add an arbitrary constant to the energy so we may assume that $V_0 = 0$ without loss of generality. Let ΔV denote the height of the potential barrier separating these minima.

If we were integrating over periodic paths, then the minimum action path would be one which simply sits at some minimum for all time, $x(\tau) = x_k$, so that $\min(S_{\text{periodic}}) = 0$. But for Z_n, we need to consider a-periodic paths. Consider first the case of $n{=}1$. Paths $\bar{x}(\tau)$ which minimize the Euclidean action must obey the variational condition $0 = \delta S$. This gives Newton's equation with an inverted potential, since we are in Euclidean space not Minkowski,

$$\frac{d^2\bar{x}}{dt^2} = V'(\bar{x})\,.$$

We want solutions which also satisfy the boundary condition for Z_1, namely $\bar{x}(\beta) = \bar{x}(0) + a$. Such a path will necessarily receive positive contributions to the action from both kinetic and potential terms. For large β, the potential contribution will be small if the path stays near some minimum, say x_k, for some portion of the time interval β, and then spends nearly all of the remaining time interval near the next minimum at x_{k+1}, with the transition between these two minima having some characteristic duration $\Delta\tau$ which is small compared to β. Such a path can be termed a "kink" configuration. Qualitatively, the potential contribution to the action, $\int d\tau\, V(x(\tau))$, will be $O(\Delta V\,\Delta\tau)$ since the path $x(\tau)$ traverses the potential barrier of height ΔV separating x_k and x_{k+1} in a time of order $\Delta\tau$. At the same time, the kinetic contribution to the action $\int d\tau\, (dx/d\tau)^2$ must be of order $\Delta\tau(a/\Delta\tau)^2 = a^2/\Delta\tau$ since the derivative $dx/d\tau$ will have an average value of order $a/\Delta\tau$ during the transition period. Minimizing the sum yields $\Delta\tau = O(a/\sqrt{\Delta V})$ and hence $S_{\min} = O(a\sqrt{\Delta V})$.

This qualitative analysis can easily be made quantitative. The above Euclidean equation of motion has as a constant of the motion the "energy" (in an inverted potential), $E \equiv \frac{1}{2}(dx/d\tau)^2 - V(x)$. In the large β limit, paths which asymptotically approach minima of the potential will have $E{=}0$, and hence kink solutions, as $\beta \to \infty$, are paths for which $dx/d\tau = \sqrt{2V(x)}$. The corresponding action is

$$S_1 \equiv \int d\tau \left[\tfrac{1}{2}\Big(\frac{dx}{d\tau}\Big)^2 + V(x)\right] = \int d\tau\; 2V(x(\tau)) = \int_{x_k}^{x_{k+1}} dx\,\sqrt{2V(x)}\,.$$

Note there is an infinite set of minimal action kink solutions due to time translation invariance — the time at which the path transitions across the barrier between minima is arbitrary.

Having established the form of $n = 1$ solutions, the $n = -1$ solution is simply a time-reversed kink, called an "antikink", which transitions between some minimum x_k and x_{k-1}. with identical action and has the same action as the kink, $S_{-1} = S_1$.

For $|n| > 1$, in the large β regime, one can construct multi-kink solutions which transition between non-adjacent minima simply by gluing together any suitable sequence of kinks and antikinks. As $\beta \to \infty$, any sequence containing N_+ kinks and N_- antikinks, with $n = N_+ - N_-$ will be a minimal action configuration contributing to Z_n. The action, is just the sum of the individual kink actions, so the minimal action configurations will consist of purely of kinks, or of antikinks, and satisfy

$$S_n = |n|\, S_1 .$$

The above is strictly correct as $\beta \to \infty$. For large but finite β there will be small corrections suppressed by $O(e^{-\beta/\Delta\tau})$.

(f) If a coupling g^2 is introduced by multiplying the Euclidean action by an overall factor of $1/g^2$, then for weak coupling, $g^2 \ll 1$, the dominant contribution to the path integral will come from small fluctuations around the relevant minimal action configurations. Single kink (or antikink) contributions will be suppressed by the exponentially small factor of e^{-S_1/g^2}. Multi-kink/antikink contributions will be suppressed by higher powers of this exponential.

To calculate a given Z_n, one must integrate over all paths with a net displacement of na. Calculating $Z(\theta) = \sum_n e^{in\theta}\, Z_n$ is then equivalent to integrating over all paths whose displacement is any integer multiple of a, $\Delta x = x(\beta) - x(0) = na$ for any $n \in \mathbb{Z}$, while weighting all contributions with a displacement of na by the phase factor $e^{in\theta}$.

For $Z_{\pm 1}$, the classical solution is a single kink (or antikink). But as noted earlier, there is a continuous family of kink solutions trivially related by time translation. This implies that

$$Z_1\big|_{\text{single-kink}} = Z_{-1}\big|_{\text{single-antikink}} = \beta K\, e^{-S_1/g^2} ,$$

where the factor of β comes from integrating over the time at which the kink occurs while K is some β-independent factor arising from the small fluctuation determinant around a given kink solution.

For Z_n with $|n| > 1$, one might think that configurations consisting only of kinks (for $n > 0$) or only antikinks (for $n < 0$) would be relevant. This would be true if the goal was to evaluate the leading behavior of $Z(\theta)$ (or Z_n) in the limit of arbitrarily small g^2, for some fixed non-infinite value of β. Of more physical interest, however, is understanding the $\beta \to \infty$ limit for some small, but fixed, value of g^2, since ground state properties are extracted from the $\beta \to \infty$ limit. The two limits, $g^2 \to 0$ and $\beta \to \infty$ do not commute, as may be seen from the above single kink contribution. If β is fixed while $g^2 \to 0$, then $\beta K e^{-S_1/g^2} \to 0$, but if g^2 is fixed while $\beta \to \infty$, then $\beta K e^{-S_1/g^2} \to \infty$.

The correct way to think about this is to view $\zeta \equiv K e^{-S_1/g^2}$ as an exponentially small (for weak coupling) probability per unit time, or *rate*, at which kinks (or antikinks) occur. Even if that rate is very small, integrating it over a sufficiently time will yield a large contribution. For very weak coupling, in the limit of large β, dominant contributions will resemble a very dilute gas of kinks and antikinks, i.e., a sequence of kinks and antikinks, each occurring randomly in time with a probability per unit time of ζ. To add up these contributions, it is simplest to calculate $Z(\theta)$ directly, instead of its individual Fourier coefficients. Including contributions from configurations consisting of N_+ kinks occurring at times $\{\tau_k\}$ and N_- antikinks occurring at times $\{\bar{\tau}_l\}$, we have

$$Z(\theta) \sim \sum_{N_+=0}^{\infty} \sum_{N_-=0}^{\infty} \frac{1}{N_+!N_-!} \int_0^\beta d\tau_1 \cdots d\tau_{N_+} \int_0^\beta d\bar{\tau}_1 \cdots d\bar{\tau}_{N_+} \, e^{i\theta(N_+-N_-)} \, \zeta^{N_++N_-} ,$$

where the factorials in the denominator compensate for what would otherwise be over-counting since all permutations of the kink times $\{\tau_k\}$, or of the antikink times $\{\bar{\tau}_l\}$, represent the same physical configuration. Performing the time integrals and then the sums is straightforward in the weak coupling regime where the kink density ζ is tiny and interactions between kinks and antikinks can be neglected. We have

$$\begin{aligned} Z(\theta) &\sim \sum_{N_+=0}^{\infty} \sum_{N_-=0}^{\infty} \frac{1}{N_+!N_-!} \, e^{i\theta(N_+-N_-)} \, (\beta\zeta)^{N_++N_-} \\ &= \sum_{N_+=0}^{\infty} \frac{1}{N_+!} \, e^{i\theta N_+} \, (\beta\zeta)^{N_+} \sum_{N_-=0}^{\infty} \frac{1}{N_-!} \, e^{-i\theta N_-} \, (\beta\zeta)^{N_-} \\ &= e^{\beta\zeta(e^{i\theta}+e^{-i\theta})} = \exp\left(2\beta K e^{-S_1/g^2} \cos\theta\right) . \end{aligned}$$

Finally, recall that the large β behavior of a partition function encodes the ground state energy via $E_0 = -\lim_{\beta\to\infty} \beta^{-1} \ln Z$. And from part d, $Z(\theta)$ is the partition function in the subspace of the Hilbert space projected onto by $\hat{P}(\theta)$. Therefore, the above large-β form for $Z(\theta)$ shows how the ground state energy in each θ-sector depends on θ,

$$E_0(\theta) \sim -2K \, e^{-S_1/g^2} \cos\theta \, .$$

One caveat is that perturbative contributions arising from fluctuations around classical solutions were not carefully accounted for in the above treatment. These contribute θ-independent factors (and the factor K should be understand as the *difference* in the perturbative contributions around a kink versus the stationary minimal path). Consequently, the ground state energy consists of θ-independent perturbative contributions plus the non-perturbative kink/antikink contribution which characterizes the effects of tunneling through the potential barriers,

$$E_0(\theta) \sim E_0^{\text{pert.}} - 2K \, e^{-S_1/g^2} \cos\theta \, .$$