1. Fundamental representation Higgs theory. Solutions by K. Aitken.

   (a) Local: $SU(N)$ gauge symmetry. This acts as

   $$\Phi \to u(x)\,\Phi, \qquad A_\mu \to u(x)\,A_\mu u(x)^\dagger + u(x)\,\partial_\mu u(x)^\dagger \tag{1}$$

   with $u(x) \in SU(N)$.
   Global: $SU(N_f = N)$ flavor symmetry, $U(1)$ particle number symmetry, and the usual spacetime symmetries. The former two can be combined to act as

   $$\Phi \to U\Phi \tag{2}$$

   with $U \in U(N_f = N)$. There is, of course, also a subgroup of the local $SU(N)$ gauge symmetries consisting of gauge transformations which happen to be constant in spacetime, and hence can be considered a global symmetry.

   (b) Let us consider each term in the action in turn. Pure gauge gauge fields minimize the Maxwell term. To see this, consider the gauge configuration $A'_\mu(x) = 0$, for which the Maxwell term $\frac{1}{2g^2}F_{\mu\nu}^2 = 0$. Similarly, to minimize the the $D_\mu\Phi$ term, consider the same gauge field configuration and also $\Phi(x) = \Phi' \equiv vI_N$. This clearly yields $D_\mu\Phi = 0$ and thus the scalar kinetic term also vanishes. Without loss of generality, we can use our gauge and global symmetries to transform this configuration into the more general forms given above

   $$\Phi' \to \Phi = v\,u(x)\,e^{i\chi}, \qquad A'_\mu \to A_\mu = u(x)\,\partial_\mu u(x)^\dagger\,, \tag{3}$$

   for arbitrary $u(x) \in SU(N)$, *provided* that the overall phase χ is constant, so that $\partial_\mu\chi(x) = 0$. Finally, consider the potential terms. Once more, without loss of generality for these vacuum configurations we can work in the gauge where $A_\mu(x) = 0$ and $\Phi(x) = \Phi'$. In this case, we have

   $$m^2\,\mathrm{tr}\Phi^\dagger\Phi = m^2 N\,v^2 \tag{4}$$

   $$\tfrac{1}{2N}\,\lambda_1\left(\mathrm{tr}\,\Phi^\dagger\Phi\right)^2 = \frac{\lambda_1}{2N}\left(Nv^2\right)^2 = \tfrac{1}{2}\,\lambda_1 N v^4\,, \tag{5}$$

   $$\tfrac{1}{2}\,\lambda_2\mathrm{tr}\left[\left(\Phi^\dagger\Phi\right)^2\right] = \tfrac{1}{2}\lambda_2\,Nv^4 = \tfrac{1}{2}\lambda_2 N v^4 \tag{6}$$

   Assuming $m^2 < 0$, this is minimized when

   $$\frac{\partial}{\partial v^2}\left[m^2Nv^2 + \tfrac{1}{2}\left(\lambda_1+\lambda_2\right)Nv^4\right] = 0 \tag{7}$$

   so that

   $$v^2 = \frac{-m^2}{\lambda_1+\lambda_2}. \tag{8}$$

   (c) Without loss of generality we can again specialize to the field configuration $\Phi'(x) = vI_N$ and consider which global symmetries leave this vacuum configuration invariant. One could perform some global flavor symmetry transformation to take $\Phi' \to U\Phi' = vU$, which seemingly changes the unbroken flavor symmetries. However, one has the global subgroup of the gauge symmetry to play with as well. By choosing the gauge transformation $V(x) = U^\dagger$, one can cancel the contribution from the global flavor transformation, meaning $\Phi' \to U\Phi' V = \Phi'$. Hence this vacuum configuration is left invariant under the "diagonal" subgroup of the $SU(N_f = N)_{\text{flavor}} \times SU(N_c = N)_{\text{gauge}}$ global symmetry. Bear in mind that since the gauge group is only $SU(N)$, there is no way to cancel an arbitrary phase transformation corresponding to the $U(1)$ global symmetry. Hence, a full $SU(N)$ global symmetry remains, but the $U(1)$ global symmetry is spontaneously broken. The spacetime symmetries remain unbroken.

(d) This decomposition is similar in spirit to polar coordinates, and is closely related to singular value decomposition. The matrix $\Phi^\dagger\Phi$ is Hermitian and positive (semi)definite. One can always diagonalize such a matrix by a unitary transformation, $\Phi^\dagger\Phi = v\lambda v^\dagger$, with $v \in U(N)$ and λ diagonal, real, and non-negative. Let $h = \sqrt{\Phi^\dagger\Phi} \equiv v\sqrt{\lambda}v^\dagger$, where $\sqrt{\lambda}$ is understood to be the positive square root. The matrix h, by construction, is Hermitian and positive (semi)definite.

If Φ is invertible, then so is $\Phi^\dagger\Phi$, and hence its eigenvalues are strictly positive. Consequently, λ is invertible and therefore so is h. Now let $w \equiv \Phi h^{-1} v$. This definition implies that $\Phi = wv^\dagger h$. The matrix w is automatically unitary since $w^\dagger w = v^\dagger h^{-1}\Phi^\dagger\Phi h^{-1} v = v^\dagger v = 1$. So $wv^\dagger$ is the product of two unitary matrices, which is also unitary. Any unitary matrix can be factored as a special unitary matrix times an overall phase, $wv^\dagger \equiv u\, e^{i\chi}$, and therefore $\Phi = u\, e^{i\chi}\, h$ as claimed, with $u \in SU(N)$, χ real, and h Hermitian and positive definite.

This decomposition is local, meaning that Φ, u, χ and h may all vary from point to point, $\Phi(x) = u(x)\, e^{i\chi(x)}\, h(x)$. Any gauge field $A_\mu(x)$ can always be written as a gauge transform of some gauge equivalent gauge field, so defining $A_\mu(x) = u(x)(\partial_\mu + a_\mu(x))u(x)^\dagger$ just means that a_μ is the gauge transformation of A_μ by the gauge transformation $u(x)^\dagger$.

(e) Let $h(x) = vI_N + \delta h(x)$, for some unspecified scalar expectation value v (which is totally unrelated to the matrix $v(x)$ above).

Starting with the kinetic terms, we have

$$\begin{aligned}
D_\mu\Phi &= (\partial_\mu + A_\mu)\Phi \\
&= \partial_\mu(u(x)e^{i\chi(x)}h(x)) + u(x)\big(\partial_\mu u(x)^\dagger + a_\mu(x)u(x)^\dagger\big)u(x)e^{i\chi(x)}h(x) \\
&= \big[(\partial_\mu u(x)) + u(x)(\partial_\mu u(x)^\dagger)u(x)\big]e^{i\chi(x)}h(x) + u(x)e^{i\chi(x)}\big[\partial_\mu + a_\mu + i\partial_\mu\chi(x)\big]h(x) \\
&= u(x)e^{i\chi(x)}\big[\partial_\mu + a_\mu + i\partial_\mu\chi(x)\big]h(x) \\
&= u(x)e^{i\chi(x)}\big[\big(a_\mu + i\partial_\mu\chi\big)vI_N + \big(\partial_\mu + a_\mu + i\partial_\mu\chi\big)\delta h\big]\,,
\end{aligned} \tag{9}$$

and thus

$$\begin{aligned}
\operatorname{tr}\big[(D_\mu\Phi)^\dagger(D_\mu\Phi)\big] &= \operatorname{tr}\big|(a_\mu + i\partial_\mu\chi)vI_N + (\partial_\mu + a_\mu + i\partial_\mu\chi)\delta h\big|^2 \\
&= Nv^2(\partial_\mu\chi)^2 + \tfrac{1}{2}v^2(a^a_\mu)^2 + \operatorname{tr}(D_\mu\delta h)^2 \\
&\quad + v\operatorname{tr}[(a_\mu + i\partial_\mu\chi)(D^\mu\delta h)^\dagger] + v\operatorname{tr}[(a_\mu - i\partial_\mu\chi)(D^\mu\delta h)^\dagger],
\end{aligned} \tag{10}$$

where we have defined a modified covariant derivative, $D_\mu\delta h \equiv (\partial_\mu + a_\mu + i\partial_\mu\chi)\delta h$, used the fact that $\partial_\mu u^\dagger = -u^\dagger(\partial_\mu u)u^\dagger$, and introduced components of the gauge field $a_\mu \equiv a^a_\mu T^a$ so that $\operatorname{tr}(a_\mu^2) = a^a_\mu a^{\mu b}\operatorname{tr}(T^aT^b) = \frac{1}{2}a^a_\mu a^{\mu a}$.

For the gauge kinetic term, the pure gauge part vanishes by construction, and thus we can simply write

$$-\tfrac{1}{2g^2}\mathrm{tr}F_{\mu\nu}^2 = -\tfrac{1}{2g^2}\mathrm{tr}f_{\mu\nu}^2 \tag{11}$$

with $f_{\mu\nu}$ the field strength for a_μ. For the potential terms, we have

$$\begin{aligned} m^2\mathrm{tr}\Phi^\dagger\Phi = m^2\mathrm{tr}[h(x)^2] &= m^2\mathrm{tr}[(vI_N + \delta h(x))^2] \\ &= m^2Nv^2 + 2m^2v\mathrm{tr}[\delta h(x)] + m^2\mathrm{tr}[\delta h(x)^2]\,. \end{aligned} \tag{12}$$

Similarly,

$$\begin{aligned} \tfrac{1}{2N}\lambda_1(\mathrm{tr}\Phi^\dagger\Phi)^2 &= \tfrac{1}{2N}\lambda_1(Nv^2 + 2v\mathrm{tr}[\delta h(x)] + \mathrm{tr}[\delta h(x)^2])^2 \\ &= \tfrac{\lambda_1}{2}Nv^4 + 2\lambda_1v^3\mathrm{tr}[\delta h(x)] + \lambda_1v^2\mathrm{tr}[\delta h(x)^2] + \tfrac{2\lambda_1}{N}v\mathrm{tr}[\delta h(x)]\mathrm{tr}[\delta h(x)^2] \\ &\quad + \tfrac{2\lambda_1}{N}v^2(\mathrm{tr}[\delta h(x)])^2 + \tfrac{\lambda_1}{2N}(\mathrm{tr}[\delta h(x)^2])^2 \end{aligned} \tag{13}$$

$$\begin{aligned} \tfrac{1}{2}\lambda_2\mathrm{tr}[(\Phi^\dagger\Phi)^2] &= \tfrac{1}{2}\lambda_2\mathrm{tr}[(vI_N + \delta h(x))^4] \\ &= \tfrac{1}{2}\lambda_2\mathrm{tr}[v^4I_N + 4v^3\delta h(x) + 6v^2\delta h(x)^2 + 4v\delta h(x)^3 + \delta h(x)^4] \\ &= \tfrac{\lambda_2}{2}Nv^4 + 2\lambda_2v^3\mathrm{tr}[\delta h(x)] + 3\lambda_2b^2\mathrm{tr}[\delta h(x)^2] + 2\lambda_2v\mathrm{tr}[\delta h(x)^3] + \tfrac{\lambda_2}{2}\mathrm{tr}[\delta h(x)^4]. \end{aligned} \tag{14}$$

All the above potential terms can be combined into a fairly messy expression which, organized by powers of δh, is the sum of:

$$\text{const:} \qquad m^2Nv^2 + \tfrac{1}{2}(\lambda_1+\lambda_2)Nv^4 = -\tfrac{1}{2}m^2Nv^2\,, \tag{15}$$

$$\delta h\text{:} \qquad 2[m^2v + (\lambda_1+\lambda_2)v^3]\mathrm{tr}[\delta h] = 0\,, \tag{16}$$

$$\delta h^2\text{:} \qquad [m^2 + \lambda_1v^2 + 3\lambda_2v^2]\mathrm{tr}[\delta h^2] + \tfrac{2\lambda_1}{N}v^2(\mathrm{tr}[\delta h])^2 = 2\lambda_2v^2\mathrm{tr}[\delta h^2] + \tfrac{2\lambda_1}{N}v^2(\mathrm{tr}[\delta h])^2\,, \tag{17}$$

$$\delta h^3\text{:} \qquad \tfrac{2\lambda_1}{N}v\mathrm{tr}[\delta h]\mathrm{tr}[\delta h^2] + 2\lambda_2v\mathrm{tr}[\delta h^3] \tag{18}$$

$$\delta h^4\text{:} \qquad \tfrac{\lambda_1}{2N}(\mathrm{tr}[\delta h^2])^2 + \tfrac{\lambda_2}{2}\mathrm{tr}[\delta h^4]\,, \tag{19}$$

where we have used $v^2 = -m^2/(\lambda_1+\lambda_2)$.

(f) The $m^2 > 0$ tree-level spectrum was:

- N^2 complex scalars, for a total of $2N^2$ degrees of freedom, with mass m.
- $N^2 - 1$ (massless) gauge bosons.

For $m^2 < 0$, we instead have:

- N^2 scalar degrees of freedom, corresponding to the components of the (Hermitian) matrix δh. The quadratic δh^2 terms do not vanish, and thus these scalars are massive. More specifically, the matrix δh can be decomposed into a trace plus a traceless part, $\delta h \equiv \frac{1}{\sqrt{N}}\,\delta\tilde{h}\,I_N + \delta h'$, with $\delta\tilde{h} \equiv \frac{1}{\sqrt{N}}\mathrm{tr}(\delta h)$, and $\mathrm{tr}\,\delta h' \equiv 0$. Since $(\mathrm{tr}[\delta h])^2 = (\delta\tilde{h})^2$, this mass term introduces a mass splitting between the traceless and trace part of δh. The traceless part has mass $m_{\delta h'}^2 \equiv 4\lambda_2v^2$ while the trace part has mass $m_{\delta\tilde{h}}^2 \equiv 4(\lambda_1+\lambda_2)v^2$.
- $N^2 - 1$ massive gauge bosons (which may be viewed as having "eaten" $N^2 - 1$ of the original scalar degrees of freedom), represented by a_μ and having mass gv.

- 1 spin-zero massless Goldstone boson arising from spontaneous breaking of the $U(1)$ global symmetry. This is represented by the above phase field χ.

Note that the counting of total degrees of freedom is identical in either case.

(g) The only interaction terms involving the Goldstone boson are produced from the terms

$$\mathrm{tr}[(D_\mu \delta h)^2] = \mathrm{tr}[(\partial_\mu + a_\mu - i\partial_\mu\chi)(\partial^\mu + a^\mu + i\partial^\mu\chi)(\delta h)^2] \tag{20}$$

$$v\mathrm{tr}[(D^\mu \delta h)^\dagger(a_\mu + i\partial_\mu\chi)] = v\mathrm{tr}[(\partial_\mu + a_\mu - i\partial_\mu\chi)\delta h^\dagger(a^\mu + i\partial^\mu\chi)] \tag{21}$$

$$v\mathrm{tr}[(a_\mu - i\partial_\mu\chi)(D^\mu\delta h)] = v\mathrm{tr}[(a_\mu - i\partial_\mu\chi)(\partial^\mu + a^\mu + i\partial^\mu\chi)\delta h]. \tag{22}$$

This produces vertices of the form

$$(\partial_\mu\chi)a(\delta h)^2, \qquad (\partial_\mu\chi)\partial_\mu(\delta h)^2, \qquad (\partial_\mu\chi)^2(\delta h)^2, \qquad (\partial_\mu\delta h)(\partial_\mu\chi), \qquad \text{etc.} \tag{23}$$

From the terms above, one sees that the Goldstone bosons do not have self-interactions — there are no interaction terms only involving the χ field. Furthermore, every interaction with other fields involves a factor of the gradient $\partial_\mu\chi$ (as opposed to just χ itself). Consequently, any scattering amplitude involving these Goldstone bosone will have at least one factor of momenta for every incoming or outgoing Goldstone boson, and squares of these factor in every cross-section. Hence, all cross-sections involving Goldstone bosons vanish as the spatial momentum (or energy) of the Goldstone boson go to zero. This is characteristic of all Goldstone bosons.

(h) As noted above, in this phase there are N^2 complex scalars and N^2-1 (perturbatively) massless gauge bosons. None of the global symmetries present are spontaneously broken in this phase, so there are no Goldstone bosons. There must be a phase transition between the $m^2 > 0$ and $m^2 < 0$ regimes since these phases have different realizations of the global symmetries. More specifically, the mass gap μ (the mass of the lightest excitation) is exactly zero in the Higgs phase due to the $U(1)$ Goldstone boson, while in the positive m^2 phase the mass gap is non-zero due to the non-Abelian interactions of the gauge bosons. No analytic function $\mu(m^2)$ can be exactly zero for $m^2 < 0$ while being non-zero for $m^2 > 0$, so there must be non-analytic behavior as m^2 varies from large negative to large positive values. In other words, there must exist at least one phase transition as m^2 varies.

(i) This term explicitly breaks the $U(1)$ global symmetry, because it prefers whatever value of the phase maximizes the real part of $\det\Phi$. Since there is no global symmetry to break, we will no longer find a Goldstone boson, and instead the associated degree of freedom will remain massive. For part (f) there will now be N^2+1 massive scalar degrees of freedom. For part (h), there is no longer a guarantee that there is any phase transition because there is no longer a sharp distinction between the two phases (either in terms of global symmetry realizations, or in the number of exactly massless excitations).

2. Adjoint representation Higgs theory. Solutions by K. Aitken.

(a) Once more there is a local $SU(N)$ gauge symmetry. Since Φ is a Hermitian scalar field, the analog of the $U(1)$ global symmetry associated with the phase is reduced to a $\mathbb{Z}_2$ global symmetry which acts as $\Phi \to -\Phi$. Once again, there is a subset of the $SU(N)$ gauge symmetry which can act globally and also the usual spacetime symmetries.

(b) We can play mostly the same game we did in Problem 1 above. First off, consider the minimization of the potential. We assume we have used our gauge freedom to diagonalize Φ and order the diagonal eigenvalues from smallest to largest. Here it will be helpful to minimize with respect to Φ^2. We have

$$\begin{aligned}\frac{\partial V}{\partial\left(\Phi_a^b\Phi_b^c\right)} &= \frac{\partial}{\partial\left(\Phi_a^b\Phi_b^c\right)}\left[\frac{1}{2}m^2\delta_c^a\Phi_a^b\Phi_b^c + \frac{1}{4N}\lambda_1\left(\delta_c^a\Phi_a^b\Phi_b^c\right)^2 + \frac{1}{4}\lambda_2\delta_e^a\Phi_a^b\Phi_b^c\Phi_c^d\Phi_d^e\right]\\ &= \frac{1}{2}m^2\delta_c^a + \frac{2}{4N}\lambda_1\delta_c^a\mathrm{tr}\left(\Phi^2\right) + \frac{2}{4}\lambda_2\Phi_c^b\Phi_b^a\\ &= 0\end{aligned} \tag{24}$$

For $a \neq c$, the first two terms vanish and thus the off diagonals Φ^2 vanish. Meanwhile, for $a = c$ we have

$$m^2 + \frac{1}{N}\lambda_1\mathrm{tr}\left(\Phi^2\right) + \lambda_2\Phi_a^b\Phi_b^a = 0 \tag{25}$$

with *no* sum over a. This implies $\Phi^2 = v^2 I_N$ with $v^2 = -m^2/\left(\lambda_1 + \lambda_2\right)$, but note this still does not specify whether the diagonal eigenvalues of Φ itself are positive or negative. Thus a general vacuum configuration is

$$\langle\Phi_0\rangle \equiv v\,\mathrm{diag}(\underbrace{-1,\ldots,-1}_{p}\underbrace{+1,\ldots+1}_{N-p}) \tag{26}$$

for $p = 0,\ldots,N$. This breaks the gauge group down to $SU\left(N\right) \to SU\left(p\right)\times SU\left(N-p\right)$ as well as breaking the $\mathbb{Z}_2$ global symmetry.

The rest of the problem follows in an identical manner similar to the previous problem. Without loss of generality, we can consider the configurations $\Phi'\left(x\right) = \langle\Phi_0\rangle$ and $A'_\mu\left(x\right) = 0$. These clearly minimize the kinetic terms. We can then use gauge and global symmetry transformations to put these in the more general forms

$$\Phi\left(x\right) = \pm u\left(x\right)\langle\Phi_0\rangle u\left(x\right)^\dagger, \qquad A_\mu\left(x\right) = u\left(x\right)\partial_\mu u\left(x\right)^\dagger. \tag{27}$$

(c) Each value of p should be examined separately. For simplicity, consider the case $p = 0$ where $\langle\Phi_0\rangle = vI_N$. Then we can repeat much of the same analysis as in the previous problem, but now none of the gauge fields are Higgsed (i.e., acquire tree-level mass). We expand using

$$\Phi\left(x\right) = \pm\, u\left(x\right)h\left(x\right)u\left(x\right)^\dagger \tag{28}$$

$$A_\mu\left(x\right) = u\left(x\right)\left(\partial_\mu + a_\mu\left(x\right)\right)u\left(x\right)^\dagger \tag{29}$$

with $h\left(d\right) = I_N + \delta h\left(x\right)$. The kinetic terms are

$$\begin{aligned}D_\mu\Phi &= \partial_\mu\Phi + \left[A_\mu, \Phi\right]\\ &= \pm u\left(x\right)\left(\partial_\mu h + \left[a_\mu, h\right]\right)u\left(x\right)^\dagger\\ &= \pm u\left(x\right)\left(\partial_\mu\delta h + \left[a_\mu, \delta h\right]\right)u\left(x\right)^\dagger\end{aligned} \tag{30}$$

and thus

$$\mathrm{tr}\left[\left(D_\mu\Phi\right)^\dagger\left(D_\mu\Phi\right)\right] = \mathrm{tr}\left|\partial_\mu\delta h + \left[a_\mu, \delta h\right]\right|^2 = \mathrm{tr}\left(D_\mu\delta h\right)^2 \tag{31}$$

where we have defined the covariant derivative $D_\mu \delta h \equiv \partial_\mu \delta h + [a_\mu, \delta h]$. Notably, the gauge field does not pick up a mass term, which is what we would expect for the choice of $\langle \Phi_0 \rangle$ above. More generally, for $0 < p < N$, the squared commutator $[a_\mu, \langle \Phi_0 \rangle]^2$ will produce mass terms for those components of the gauge field which lie outside the unbroken (gauge) symmetry group.

Once more, for the gauge kinetic term the pure gauge part vanishes by construction, and thus we can simply write

$$-\frac{1}{2g^2}\mathrm{tr}F_{\mu\nu}^2 = -\frac{1}{2g^2}\mathrm{tr}f_{\mu\nu}^2 \tag{32}$$

with $f_{\mu\nu}$ the field strength for a_μ. For the potential terms, we have

$$\begin{aligned}\frac{1}{2}m^2\mathrm{tr}\Phi^2 &= \frac{1}{2}m^2\mathrm{tr}\left[h(x)\,h(x)\right] \\ &= \frac{1}{2}m^2Nv^2 + m^2v\mathrm{tr}\left[\delta h(x)\right] + \frac{1}{2}m^2\mathrm{tr}\left[\delta h(x)^2\right].\end{aligned} \tag{33}$$

Similarly,

$$\begin{aligned}\frac{1}{4N}\lambda_1\left(\mathrm{tr}\Phi^2\right)^2 &= \frac{1}{4N}\lambda_1\left(Nv^2 + 2v\mathrm{tr}\left[\delta h(x)\right] + \mathrm{tr}\left[\delta h(x)^2\right]\right)^2 \\ &= \frac{\lambda_1}{4}Nv^4 + \lambda_1 v^3\mathrm{tr}\left[\delta h(x)\right] + \frac{\lambda_1}{4}v^2\mathrm{tr}\left[\delta h(x)^2\right] + \frac{\lambda_1}{N}v\mathrm{tr}\left[\delta h(x)\right]\mathrm{tr}\left[\delta h(x)^2\right] \\ &+ \frac{\lambda_1}{N}v^2\left(\mathrm{tr}\left[\delta h(x)\right]\right)^2 + \frac{\lambda_1}{4N}\left(\mathrm{tr}\left[\delta h(x)^2\right]\right)^2\end{aligned} \tag{34}$$

$$\begin{aligned}\frac{1}{4}\lambda_2\mathrm{tr}\left[\Phi^4\right] &= \frac{1}{4}\lambda_2\mathrm{tr}\left[\left(vI_N + \delta h(x)\right)^4\right] \\ &= \frac{1}{4}\lambda_2\mathrm{tr}\left[v^4I_N + 4v^3\delta h(x) + 6v^2\delta h(x)^2 + 4v\delta h(x)^3 + \delta h(x)^4\right] \\ &= \frac{\lambda_2}{4}Nv^4 + \lambda_2 v^3\mathrm{tr}\left[\delta h(x)\right] + \frac{3}{4}\lambda_2 b^2\mathrm{tr}\left[\delta h(x)^2\right] + \lambda_2 v\mathrm{tr}\left[\delta h(x)^3\right] + \frac{\lambda_2}{4}\mathrm{tr}\left[\delta h(x)^4\right].\end{aligned} \tag{35}$$

(d) The tree-level spectrum obviously depends on the value of p above. The general case which breaks $SU(N) \to SU(p) \times SU(N-p)$ leaves the $SU(p)$ and $SU(N-p)$ subgroups unbroken. The remaining gauge bosons degrees of freedom eat the corresponding scalars to become massive. There is no massless Goldstone boson because the spontaneously broken $\mathbb{Z}_2$ symmetry is not continuous.

(e) For $m^2 > 0$ all the scalar degrees of freedom are massive with mass m and the gauge bosons are (perturbatively) massless. The $\mathbb{Z}_2$ global symmetry is unbroken. Since the $\mathbb{Z}_2$ symmetry realization changes between spontaneously broken and unbroken as m^2 varies from large negative to large positive values, there must be a phase transition between the two regimes.