Consistency

Definition: A set Σ of formulas is *consistent* iff there is some formula ψ such that $\Sigma \nvdash \psi$. (I.e., Σ is consistent iff not everything is derivable from Σ.)

Theorem. Σ is consistent iff for no formula φ do we have both $\Sigma \vdash \varphi$ and $\Sigma \vdash \sim\varphi$.
(Equivalently, Σ is inconsistent iff for some φ both $\Sigma \vdash \varphi$ and $\Sigma \vdash \sim\varphi$.)

Proof.

If both $\Sigma \vdash \varphi$ and $\Sigma \vdash \sim\varphi$, then we will show that Σ is inconsistent. For *any* formula ψ, the formula $[\varphi \rightarrow (\sim\psi \rightarrow \varphi)]$ is a theorem (e.g., by Axiom 1). Thus $\Sigma \vdash [\varphi \rightarrow (\sim\psi \rightarrow \varphi)]$. If $\Sigma \vdash \varphi$, then by Modus Ponens (MP) $\Sigma \vdash (\sim\psi \rightarrow \varphi)$. If also $\Sigma \vdash \sim\varphi$, then by Modus Tollens (MT) $\Sigma \vdash \sim\sim\psi$. By Double Negation (DN), $\Sigma \vdash \psi$. So Σ is inconsistent.

Conversely, if Σ is inconsistent then every formula is derivable from Σ, in particular, some formula φ and its negation $\sim\varphi$.

The Basic Connection (TBC) (*syntactic*)

TBC : $\Sigma \vdash \varphi$ iff $(\Sigma \cup \{\sim\varphi\})$ is inconsistent.

Proof.

1. Suppose that $\Sigma \vdash \varphi$. Then $(\Sigma \cup \{\sim\varphi\}) \vdash \varphi$ and since $(\Sigma \cup \{\sim\varphi\}) \vdash \sim\varphi$, it follows that $(\Sigma \cup \{\sim\varphi\})$ is inconsistent.
2. Suppose that $(\Sigma \cup \{\sim\varphi\})$ is inconsistent. Then everything can be derived from $(\Sigma \cup \{\sim\varphi\})$. In particular if τ is a theorem (e.g., maybe τ is "(P$\rightarrow$P)") then $(\Sigma \cup \{\sim\varphi\}) \vdash \sim\tau$. By the Deduction Theorem, $\Sigma \vdash (\sim\varphi \rightarrow \sim\tau)$. But since τ is a theorem, by DN, we have $\vdash \sim\sim\tau$ and so $\Sigma \vdash \sim\sim\tau$. Then by MT, $\Sigma \vdash \sim\sim\varphi$. Hence $\Sigma \vdash \varphi$.

Note that (by DN) another form of TBC would be

$$\Sigma \vdash \sim\varphi \text{ iff } (\Sigma \cup \{\varphi\}) \text{ is inconsistent.}$$

Also note that when Σ is empty TBC characterizes theorems:

$$\vdash \varphi \text{ iff } \{\sim\varphi\} \text{ is inconsistent.}$$

Homework Problem: Carry out part (2) of the TBC proof using MP and Axiom 3 $[\vdash (((\sim\alpha \rightarrow \sim\beta) \rightarrow (\sim\alpha \rightarrow \beta)) \rightarrow \alpha)]$ instead of MT and DN.