Some General Concepts

I will assume the intuitive notion of an "effective procedure".

Defn: A set ∑ of expressions is DECIDABLE iff there is an effective procedure such that, given a putative member x of ∑ , the procedure will decide whether or not x actually *is* in ∑.

Question: how many effective procedures are there?
Answer: Countably many. For each procedure must be finitely describable and there are only countably many finite sequences.

Example.
(1) In any formal system the set of well formed expressions is decidable.
(2) In any formal system the set of proofs (or derivations) is decidable.
(3) The set of tautologies of sentential logic is decidable. Truth tables do it. Given the completeness theorem, it follows that the set of theorems is decidable.

Defn: A formal system is decidable iff the set of its theorems is decidable.

So sentential logic is decidable.
Monadic logic is decidable. (Need only look at models of size 2^n, where n is the number of distinct predicate and sentence letters.)
But polyadic logic is not decidable (Church's Theorem). Indeed the logic of 2-place predicates is not decidable.

Defn: A set ∑ is effectively enumerable (e/e) iff there is some effective procedure that generates all of ∑.

Theorem: If the putative members of ∑ are (e/e) , then if ∑ is decidable, it is e/e too.
Proof. Effectively enumerate all the putative members and decide for each one whether it belongs to ∑!

Theorem. If both ∑ and its complement ∑' are e/e, then ∑ is decidable.
Proof. Consider any putative member x of ∑. We can begin an enumeration of ∑ and of ∑' in tandem. Then, after finitely many steps x will turn up in one of those lists, which decides the case.
Cor. In any FS, if the non-theorems are e/e, the system is decidable.

Defns.
In any formal system a set ∑ of formulas is deductively closed iff $\Sigma \vdash \phi$ implies that $\phi \in \Sigma$.
A formal system FS (with negation) is *negation complete* iff for every closed formula of the system either $FS \vdash \phi$ or $FS \vdash \sim\phi$.

Fact. Sentential logic is not negation complete. E.G., neither P nor ~P is a theorem. Thus in general predicate logic is not negation complete.

Theorem. If Σ is consistent, e/e, deductively closed and negation complete, then Σ is decidable.
Proof. Since Σ is e/e it follows that all the consequences (under $\vdash$) of Σ are also e/e. (We need to prove this!) Let ϕ be any formula of the system. We want to decide whether or not $\phi \in \Sigma$. Since Σ is deductively closed it is sufficient to determine whether $\Sigma \vdash \phi$. Since Σ is negation complete in the e/e enumeration of the consequences of Σ either ϕ or $\sim\phi$ will turn up after a finite stretch, and not both since Σ is consistent. This gives us an effective procedure for deciding whether or not ϕ belongs to Σ.

Cor. If a consistent formal system is undecidable, then it is negation incomplete; i.e., there is a closed ϕ such that neither $\Sigma \vdash \phi$ not $\Sigma \vdash \sim\phi$.