EXERCISES

2.1. Explain why $2 \in \{1, 2, 3\}$.

2.2. Is $\{1, 2\} \in \{\{1, 2, 3\}, \{1, 3\}, 1, 2\}$? Justify your answer.

2.3. Try to devise a set which is a member of itself.

2.4. Give an example of sets A, B, and C such that $A \in B$, $B \in C$, and $A \notin C$.

2.5. Describe in prose each of the following sets.

(a) $\{x \in \mathbb{Z} \mid x$ is divisible by 2 and x is divisible by $3\}$.

(b) $\{x \mid x \in A$ and $x \in B\}$.

(c) $\{x \mid x \in A$ or $x \in B\}$.

(d) $\{x \in \mathbb{Z}^+ \mid x \in \{x \in \mathbb{Z} \mid$ for some integer y, $x = 2y\}$ and $x \in \{x \in \mathbb{Z} \mid$ for some integer y, $x = 3y\}\}$.

(e) $\{x^2 \mid x$ is a prime$\}$.

(f) $\{a/b \in \mathbb{Q} \mid a + b = 1$ and $a, b \in \mathbb{Q}\}$.

(g) $\{\langle x, y\rangle \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.

(h) $\{\langle x, y\rangle \in \mathbb{R}^2 \mid y = 2x$ and $y = 3x\}$.

2.6. Prove that if a, b, c, and d are any objects, not necessarily distinct from one another, then $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$ iff both $a = c$ and $b = d$.

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3.1. Prove each of the following, using any properties of numbers that may be needed.

(a) $\{x \in \mathbb{Z} \mid \text{for an integer } y, x = 6y\} = \{x \in \mathbb{Z} \mid \text{for integers } u \text{ and } v, x = 2u$ $\text{and } x = 3v\}$.

(b) $\{x \in \mathbb{R} \mid \text{for a real number } y, x = y^2\} = \{x \in \mathbb{R} \mid x \geq 0\}$.

(c) $\{x \in \mathbb{Z} \mid \text{for an integer } y, x = 6y\} \subseteq \{x \in \mathbb{Z} \mid \text{for an integer } y, x = 2y\}$

3.2. Prove each of the following for sets A, B, and C.

(a) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

(b) If $A \subseteq B$ and $B \subset C$, then $A \subset C$.

(c) If $A \subset B$ and $B \subseteq C$, then $A \subset C$.

(d) If $A \subset B$ and $B \subset C$, then $A \subset C$.

3.3. Give an example of sets A, B, C, D, and E which satisfy the following conditions simultaneously: $A \subset B$, $B \in C$, $C \subset D$, and $D \subset E$.

3.4. Which of the following are true for all sets A, B, and C?

(a) If $A \notin B$ and $B \notin C$, then $A \notin C$.

(b) If $A \neq B$ and $B \neq C$, then $A \neq C$.

(c) If $A \in B$ and $B \not\subseteq C$, then $A \notin C$.

(d) If $A \subset B$ and $B \subseteq C$, then $C \not\subseteq A$.

(e) If $A \subseteq B$ and $B \in C$, then $A \notin C$.

3.5. Show that for every set A, $A \subseteq \varnothing$ iff $A = \varnothing$.

3.6. Let $A_1, A_2, \cdots, A_n$ be n sets. Show that

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq A_1 \quad \text{iff} \quad A_1 = A_2 = \cdots = A_n.$$

3.7. Give several examples of a set X such that each element of X is a subset of X.

3.8. List the members of $\mathcal{P}(A)$ if $A = \{\{1, 2\}, \{3\}, 1\}$.

3.9. For each positive integer n, give an example of a set A_n of n elements such that for each pair of elements of A_n, one member is an element of the other

$\{\varnothing\}$, $\{\varnothing, \{\varnothing\}\} - \{\{\varnothing\}\}$.

4.6. Suppose A and B are subsets of U. Show that in each of (a), (b), and (c) below, if any one of the relations stated holds, then each of the others holds.

(a) $A \subseteq B$, $\overline{A} \supseteq \overline{B}$, $A \cup B = B$, $A \cap B = A$.

(b) $A \cap B = \varnothing$, $A \subseteq \overline{B}$, $B \subseteq \overline{A}$.

(c) $A \cup B = U$, $\overline{A} \subseteq B$, $\overline{B} \subseteq A$.

6.2. Write the members of $\{1, 2\} \times \{2, 3, 4\}$. What are the domain and range of this relation? What is its graph?

6.3. State the domain and the range of each of the following relations, and then draw its graph.

(a) $\{\langle x, y\rangle \in \underline{R} \times \underline{R} \mid x^2 + 4y^2 = 1\}$.

(b) $\{\langle x, y\rangle \in \underline{R} \times \underline{R} \mid x^2 = y^2\}$.

(c) $\{\langle x, y\rangle \in \underline{R} \times \underline{R} \mid |x| + 2|y| = 1\}$.

(d) $\{\langle x, y\rangle \in \underline{R} \times \underline{R} \mid x^2 + y^2 < 1 \text{ and } x > 0\}$.

(e) $\{\langle x, y\rangle \in \underline{R} \times \underline{R} \mid y \geq 0 \text{ and } y \leq x \text{ and } x + y \leq 1\}$.

6.4. Write the relation in Exercise 6.3(c) as the union of four relations and that in Exercise 6.3(e) as the intersection of three relations.

6.5. The formation of the cartesian product of two sets is a binary operation for sets. Show by examples that this operation is neither commutative nor associative.

7.6. Give an example of these relations.

(a) A relation which is reflexive and symmetric but not transitive in some set.

(b) A relation which is reflexive and transitive but not symmetric in some set.

(c) A relation which is symmetric and transitive but not reflexive in some set.

8.4. Using only mappings of the form $f: \underline{Z}^+ \to \underline{Z}^+$, give an example of a function which

(a) is one-to-one but not onto;

(b) is onto but not one-to-one.

8.5. Let $A = \{1, 2, \cdots, n\}$. Prove that if a map $f: A \to A$ is onto, then it is one-to-one, and that if a map $g: A \to A$ is one-to-one, then it is onto.