Interpretations and Models

An *interpretation* provides a specific domain of quantification (the universe) and assigns to the predicates specific relational properties over that domain; thus an interpretation gives us the extensions of our terms and provides truth conditions for all the closed formulas (sentences). A *model* is an interpretation that makes all our sentences true.

Formally, let Σ be a set of sentences of predicate logic (i.e., closed formulas -- no free variables). By the vocabulary of Σ we mean all the name letters (NL), sentence letters (SL) and predicate letters (PL) occurring in any of the sentence in Σ, plus *any* variable. We assume that the universe **U** of an interpretation is never empty.

An *interpretation* ***I*** of Σ in universe **U** is a function on the vocabulary of Σ (excluding the variables) whose values are called "extensions", defined as follows.

I(NL in vocab Σ) is an element in the universe **U**.

I(SL in vocab Σ) is either **U** or the empty set Φ.

I(n-place PL in vocab Σ) is a subset of the n-fold Cartesian product of **U**, $\mathbf{U}^n$.

Truth relative to an interpretation.

We want to define truth values (i.e., a valuation function val_I) relative to an interpretation **I**. So proceed as follows.

1. $val_I(SL) = \begin{cases} \text{True, if I(SL)} = \text{U} \\ \text{False, if I(SL)} = \Phi \end{cases}$
2. $val_I[\text{F}^n(\text{A}_1,\text{A}_2,\ldots,\text{A}_n)] = \text{T iff } \langle \text{I}(\text{A}_1),\text{I}(\text{A}_2),\ldots,\text{I}(\text{A}_n)\rangle \in \text{I}(\text{F}^n)$ for any n-place predicate F^n and name letters $\text{A}_1,\text{A}_2,\ldots,\text{A}_n$.

3. $val_I(\sim\varphi) = \begin{cases} \text{T, if } val_I(\varphi) = \text{F} \\ \text{F, if } val_I(\varphi) = \text{T} \end{cases}$
4. $val_I(\varphi \to \psi) = \text{T} \quad \text{iff} \quad val_I(\varphi) = \text{F} \ \text{or}\ val_I(\psi) = \text{T}$

To get to quantification we extend the vocabulary of Σ by adding a new name letter 't' not occurring in any sentence of Σ. For $u \in \mathbf{U}$, by $\mathbf{I}^t_u$ we mean an interpretation that agrees with **I** on the vocabulary of Σ, and assigns to the new name letter 't' the element u of **U**:

$$\mathbf{I}^t_u(t) = u \text{ and } \mathbf{I}^t_u(\gamma) = \mathbf{I}(\gamma) \text{ for } \gamma \in \text{vocabulary } \Sigma \text{ and } \gamma \neq t.$$

If $\phi(\alpha)$ is a formula with at most the one free variable α and 't' is the new name letter, then write $\phi(\alpha/t)$ for the result of replacing all free occurrences (if any) of the variable α in $\phi(\alpha)$ with the name letter 't'.

Examples. If $\phi(x)$ is '(ΛyFxy $\to$ ΛxGx)', then $\phi(x/t)$ is 'ΛyFty $\to$ ΛxGx)'. If $\phi(x)$ is 'ΛxFAx', then $\phi(x/t)$ is 'ΛxFAx' [note here that 'x' is not free in $\phi(x)$].

Then we define $\underset{I}{\text{val}}\,[\Lambda\alpha\phi(\alpha)]$ as follows:

$$\underset{I}{\text{val}}\,[\Lambda\alpha\phi(\alpha)] = \begin{cases} \text{T iff } \forall u \in \mathbf{U}, \underset{\mathbf{I}_u^t}{\text{val}}\,\phi(\alpha / t) = \text{T} \\ \text{F iff } \exists u \in \mathbf{U}, \underset{\mathbf{I}_u^t}{\text{val}}\,\phi(\alpha / t) = \text{F} \end{cases}$$

Examples.

1. $\underset{I}{\text{val}}\,[\Lambda x Fx] = \text{T}$ iff $\forall u \in \mathbf{U}$, $\underset{\mathbf{I}_u^t}{\text{val}}\,Ft = \text{T}$,
   iff $\forall u \in \mathbf{U}$, $\mathbf{I}_u^t(t) \in \mathbf{I}_u^t(F)$,
   iff $\forall u \in \mathbf{U}$, $u \in \mathbf{I}\,(F)$.

2. $\underset{I}{\text{val}}\,[\Lambda x \Lambda y\ G(xy)] = \text{T}$ iff $\forall u \in \mathbf{U}$, $\underset{\mathbf{I}_u^t}{\text{val}}\,\Lambda y\ G(t\,y) = \text{T}$,
   iff $\forall u \in \mathbf{U}\ \&\forall u' \in \mathbf{U}$, $\underset{\mathbf{I}_{u,u'}^{t,t'}}{\text{val}}\,G(t\,t') = \text{T}$,
   iff $\forall u \in \mathbf{U}\ \&\forall u' \in \mathbf{U}$, $< \mathbf{I}_{uu'}^{tt'}(t), \mathbf{I}_{uu'}^{tt'}(t') > \in \mathbf{I}_{uu'}^{tt'}(G^2)$
   iff $\forall u \in \mathbf{U}\ \&\forall u' \in \mathbf{U}$, $<u, u'> \in \mathbf{I}(G^2)$

Problems:

1. Prove that $\underset{I}{\text{val}}\,[V\alpha\phi(\alpha)] = \begin{cases} \text{T iff } \exists u \in \mathbf{U}, \underset{\mathbf{I}_u^t}{\text{val}}\,\phi(\alpha / t) = \text{T} \\ \text{F iff } \forall u \in \mathbf{U}, \underset{\mathbf{I}_u^t}{\text{val}}\,\phi(\alpha / t) = \text{F} \end{cases}$

2. Show that (a) $\underset{I}{\text{val}}\,[VxFx] = \text{T}$ iff $\exists u \in \mathbf{U}$ such that $u \in \mathbf{I}\,(F)$;
   (b) $\underset{I}{\text{val}}\,[VxVy\ G(xy)] = \text{T}$ iff $\exists u \in \mathbf{U}\ \&\exists u' \in \mathbf{U}$ such that $<u, u'> \in \mathbf{I}(G^2)$.

3. Write out the truth conditions for $\Lambda x V y \Lambda z\ F(xyx)$.

Models, Satisfiability and Semantic Consequence

An interpretation $\mathbf{I}$ of a set Σ of sentences is a *model* for Σ just in case $val_I(\varphi) = \text{T}, \forall \varphi \in \Sigma$.

We say that Σ is *satisfiable* iff Σ has a model.

We write $\Sigma \models \varphi$ iff every interpretation of $\Sigma \cup \{\varphi\}$ that is a model of Σ is also model of $\{\varphi\}$.