Part 2. Making a Model

Lemma 3. Every reliable, maximally consistent set of sentences has a model.
Proof. Let Δ be a reliable, maximally consistent set of sentences. Set

$$U=\{\text{name letters in the vocabulary of } \Delta\}.$$

We proceed to define an interpretation I whose universe is U.

a) $I(\alpha)=\alpha$,
   for every name $\alpha \in$ vocabulary of Δ.

b) $I(P)=U$ iff $P \in \Delta$; otherwise $I(P)=\emptyset$,
   for every sentence letter *P*.

c) $I(F^n)=\{<\alpha_1,\alpha_2, \ldots, \alpha_n> \in U^n | 'F^n\ \alpha_1,\alpha_2, \ldots, \alpha_n\ ' \in \Delta\}$,
   for every n-place predicate letter F^n; i.e.,

the n-tuple of names $<\alpha_1,\alpha_2, \ldots, \alpha_n>$ is in the extension $I(F^n)$ of an n-place predicate letter F^n iff the sentence "$F^n\ \alpha_1,\alpha_2, \ldots, \alpha_n$" is itself among the sentences of Δ.

We now need to show that every sentence in Δ comes out true under I; i.e., that

$$val_I(\phi)=T \text{ for every } \phi \in \Delta.$$

Actually we will show something stronger; namely, that $\Delta=\{\phi | val_I(\phi)=T\}$; that is, that

$$val_I(\phi)=T \text{ iff } \phi \in \Delta.$$

Our procedure here follows the technique used in the truth lemma in the case of sentential logic. So we introduce a "happiness" predicate ☺ defined for every sentence Ψ in the vocabulary of Δ by

$$☺(\Psi)\text{: } val_I(\Psi)=T \text{ iff } \Psi \in \Delta.$$

Our aim is to show that our I-interpretation makes every Ψ happy!

We begin with sentence letters and predicate letters applied to appropriate strings of names. Once we show that all of these are happy we move on first to sentential combinations of happy sentences, and then to quantification. In each case we show that happiness spreads out: if you start with happy ones you wind up with happy ones.

1. Sentence letters. Recall that for a sentence letter *P*, $I(P)=U$ iff $P \in \Delta$. Since $val_I(P)=T$ iff $I(P)=U$, we get $val_I(P)=T$ iff $P \in \Delta$.
2. Predicate letters. Recall that the n-tuple of names $<\alpha_1,\alpha_2, \ldots, \alpha_n>$ is in the extension $I(F^n)$ of an n-place predicate letter F^n iff the sentence "$F^n\ \alpha_1,\alpha_2, \ldots, \alpha_n$" is itself among the sentences of Δ.
But $val_I(F^n\ \alpha_1,\alpha_2, \ldots, \alpha_n)=T$ iff
$<\alpha_1,\alpha_2, \ldots, \alpha_n> \in I(F^n)$. So
$val_I(F^n\ \alpha_1,\alpha_2, \ldots, \alpha_n)=T$ iff
$F^n\ \alpha_1,\alpha_2, \ldots, \alpha_n \in \Delta$.

3. Negation. We want that ☺(Ψ) $\Rightarrow$ ☺($\sim\Psi$).
4. Conditionalization. We want that if ☺(Ψ_1) and ☺(Ψ_2) then ☺($\Psi_1 \rightarrow \Psi_2$).
These cases follow exactly as in the sentential truth lemma. (*Make sure; work them out for homework.*)

5. Quantification. Here we have some work to do. So I'll use a lemma as follows.

Lemma. If 'x' is the only variable free in $\phi(x)$, then $\bigwedge x\phi(x) \in \Delta$ iff $\phi(\alpha) \in \Delta$ for all name letters α in the vocabulary of Δ.

Proof of the lemma.

$\Rightarrow$ Suppose that $\bigwedge x\phi(x) \in \Delta$. By universal instantiation, $\bigwedge x\phi(x) \vdash \phi(\alpha)$ for all name letters α in the vocabulary of Δ. Since Δ is maximally consistent it follows that $\phi(\alpha) \in \Delta$ for all name letters α in the vocabulary of Δ.

$\Leftarrow$ To show the converse, suppose that

$\bigwedge x\phi(x) \notin \Delta$. Then, by maximality,

$\sim\bigwedge x\phi(x) \in \Delta$. So $\Delta \vdash \sim\bigwedge x\, \phi(x)$. Since Δ is reliable we must have that $\Delta \vdash \sim \phi(\beta)$ for some name letter β in the vocabulary of Δ. Since Δ is consistent it follows that $\phi(\beta) \notin \Delta$. Hence if $\bigwedge x\phi(x) \notin \Delta$ then it is not the case that $\phi(\alpha) \in \Delta$ for all name letters α in the vocabulary of Δ.

Ok, now, suppose that 'x' is the only variable free in $\phi(x)$. We will assume that for any name letter α in the vocabulary of Δ, ☺($\phi(\alpha)$). We want to show that ☺($\bigwedge x\phi(x)$); that is, we want $val_I(\bigwedge x\phi(x))=T$ iff $\bigwedge x\phi(x) \in \Delta$.

But $val_I(\bigwedge x\phi(x))=T$ iff $val_J(\phi(x/t))=T$

under all interpretations $J=I^t_\alpha$; that is, for every $\alpha \in U$;

iff $val_I(\phi(\alpha)) = T$ for every $\alpha \in U$; iff $\phi(\alpha) \in \Delta$, for every $\alpha \in U$, by☺($\phi(\alpha)$);

iff $\bigwedge x\phi(x) \in \Delta$, by the lemma.

(Recall that $U=\{$name letters α in the vocabulary of $\Delta\}$.)

To complete the proof just note that every sentence in the vocabulary of Δ is built up from the sentence letters and n-place predicate letters applied to n names, by taking negations , or conditionalizing, or substituting a variable for a name letter and quantifying the occurrence of that variable universally. Hence the five parts here cover all ways of forming sentences. Thus☺(Ψ) for every sentence Ψ in the vocabulary of Δ; that is, $val_I(\Psi)=T$ iff $\Psi \in \Delta$.

It follows that I is a model of Δ, so Δ is satisfiable.

Summary and Some Consequences

Lemma 1. If Σ is a consistent set of sentences then $\Sigma \subseteq \Sigma'$ where Σ' is consistent and reliable. ***Lemma 2***. If Σ' is consistent and reliable then there is a maximally consistent set Δ such that $\Sigma' \subseteq \Delta$, and Δ is also reliable. ***Lemma 3***. Every reliable, maximally consistent set Δ of sentences has a model.

***Theorem* (*Anti-Existentialist*)**. Every consistent set of sentences in pure predicate logic has a model.

***Corollary 1* (*Löwenheim-Skolem Theorem*).** Every consistent set of sentences has a model in a universe whose cardinality is either finite or denumerably infinite (a countable model).

Corollary 2 (Completenesss theorem). $\Sigma \vDash \phi \Rightarrow \Sigma \vdash \phi$.

***Corollary 3* (*Compactness Theorem*)**. If Σ is a set of sentences and every finite subset of Σ has a model then Σ has a model.