Theorem: Every consistent set of sentences has a model.

Technique of proof. We'll extend any consistent set of sentences to a reliable, maximally consistent set, which has a model.

Part 1. Extending the consistent set.

Defn. A set $\sum$ of sentences is *RELIABLE* iff whenever the negation of a universal sentence is derivable from $\sum$, so is a "witness' – a specific counter instance. In symbols.

$$\sum\vdash\sim\bigwedge x\phi(x) \Rightarrow \sum\vdash\sim\phi(\beta)$$

for some name letter β ***in the vocabulary of*** $\sum$.

Here we call β a "witness" to the failure of the universal $\bigwedge x\phi(x)$. This might look like the rule we call "existential instantiation", except that here the instance β must be among the names in the vocabulary of $\sum$.

Lemma 1. If $\sum$ is consistent then $\sum\subseteq\sum'$ where $\sum'$ is consistent and reliable.

So we'll assume that $\sum$ is consistent.The idea of the proof here is to extend $\sum$ by adding to it all conditionals whose antecedents deny a universal and whose consequents witness that denial. If we can stuff $\sum$ this way we will force it to be reliable. But will it remain consistent? The devil is in the details but the answer is "yes" if we do it carefully. Here's how.

Step 1. Begin by adding to the vocabulary of $\sum$ a countably infinite set of new name letters.

Step 2. In this new vocabulary all the well formed formulas are effectively enumerable. So we can enumerate all the formulas with at most one free variable; call them

$\phi_1(\alpha_1), \phi_2(\alpha_2), \ldots$

where α_n is the variable free in $\phi_2(\alpha_2)$ (if any are).

Step 3. Look at the list of new name letters. Among them let β_1 be the first of the new names that does not occur in the formula $\phi_1(\alpha_1)$. In general let β_n be the first new name that is different from all the previous βs and that does not occur in $\phi_i(\alpha_i)$ for $i\leq n$.

Step 4. Here we will introduce a lot of conditionals Ψ_n , to be added to $\sum$.

$$\Psi_n :(\sim \bigwedge\alpha_n\phi_n(\alpha_n)\rightarrow \sim\phi_n(\beta_n)).$$

Notice that for a given n, β_n is a name that does not occur in $\sum$ nor in any of the Ψ_m that precede it.

Step 5. If we add any finite number of Ψ_ns to Σ the result is consistent. We'll prove this by induction on n.
For n=1. We want to show that $\Sigma\cup\{\Psi_1\}$ is consistent. If not then $\Sigma\vdash\sim \Psi_1$. Then (i) $\Sigma\vdash\sim\bigwedge\alpha_1\phi_1(\alpha_1)$ and (ii) $\Sigma\vdash\phi_1(\beta_1)$. But β_1 is a name that does not occur in any formula in Σ. Hence from (ii) (by "universal derivation") $\Sigma\vdash\bigwedge\alpha_1\phi_1(\alpha_1)$, contradicting (i). Thus $\Sigma\cup\{\Psi_1\}$ is consistent.

For n=m, we can assume that
$\Sigma\cup\{\Psi_1, \ldots, \Psi_m\}$is consistent. Then
$\Sigma\cup\{\Psi_1, \ldots, \Psi_m,\Psi_{m+1}\}$is consistent. For otherwise, $\Sigma\cup\{\Psi_1, \ldots, \Psi_m\}\vdash\sim \Psi_{m+1}$. As in the n=1 case, this would imply that
$\Sigma\cup\{\Psi_1, \ldots, \Psi_m\}$ was inconsistent. (***You should show this!***)

Thus for all n, $\Sigma\cup\{\Psi_1, \ldots, \Psi_n\}$ is consistent.

Step 6. Here finally we identify the Σ' which extends Σ consistently and reliably. Just let $\Sigma'=\Sigma\cup\{\Psi_1, \Psi_2, \ldots\}$. If Σ' were inconsistent that would show up in some finite stretch $\Sigma\cup\{\Psi_1, \ldots, \Psi_n\}$, But we have just shown (step 5) that all of these are consistent. So Σ' is consistent and clearly $\Sigma\subseteq\Sigma'$. But just as clearly, Σ' is reliable. For if $\Sigma'\vdash\sim\bigwedge x\phi(x)$ then a conditional $(\sim\bigwedge x\phi(x)\longrightarrow\sim\phi(\beta))$ is already in Σ', hence we get $\Sigma'\vdash\sim\phi(\beta)$. And here β is a name in the vocabulary of Σ' just as reliability requires.

This (whew!) proves lemma 1. It now remains to extend Σ'.

Lemma 2. If Σ' is consistent and reliable then there is a maximally consistent set Δ such that $\Sigma'\subseteq\Delta$, and Δ is also reliable.

Proof. Use the old procedure on sentences in the vocabulary of Σ' to extend Σ' to a maximally consistent Δ. This process does not alter the vocabulary; it adds new sentences to Σ' but takes nothing away. Hence Δ will still contain all conditionals whose antecedents deny a universal and whose consequents contain a witness to that denial -- in the vocabulary of Δ. That is, Δ will be reliable.

Given the two lemmas: If Σ is a consistent set of sentences then there is a maximally consistent and reliable set Δ of sentences such that $\Sigma\subseteq\Delta$.

It remains to show that Δ has a model, which is Part 2.