A Beginner's Guide to the Quantum Formalism

Physics	Mathematics
1. QM systems	State Space (in general an infinite dimensional Hilbert space) over the complex numbers
2. State of the system	State vector (unit vector (i.e. length 1) in the state space). Other terms: "state function", "psi-function"
3. Observables (i.e., measurable quantities)	"Nice" (self adjoint) linear operators on the state space
4. Possible values for an observable A	Eigenvalues of the operator that corresponds to A. "Nice" in (3) above implies that these eigenvalues are real numbers.
5. Time evolution of states for an isolated system	The change from initial to final state is determined by solving the (time dependent) Schrödinger equation for the system. This can be written in the form $\psi(t) = \mathbf{H}\psi(0)$, where $\psi(t)$ and $\psi(0)$ are, respectively, the final and initial states and where **H** is a "total energy" operator (the Hamiltonian).
6. Probabilities	Born's Rule (due to Max Born) below

Born's Rule. If the state of the system is the superposition

$\psi = c_1\varphi_1 + c_2\varphi_2 + \ldots$, where

$\varphi_1, \varphi_2, \ldots$ are all the eigenstates of an observable A, corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots$ of A, then the probability $\mathrm{Pr}ob_\psi(A = \lambda_n)$ of finding that A has the value λ_n in state ψ is given by $|c_n|^2$.

Corollary 1. The average or expected value $\langle A \rangle_\psi$ of A in state ψ is given by

$$\langle A \rangle_\psi = |c_1|^2 \lambda_1 + |c_2|^2 \lambda_2 + \ldots .$$

Corollary 2. In an eigenstate of A the probability is 1 for finding the corresponding eigenvalue of A.

Paul Dirac introduced what is called the "bra" and "ket" notation. Here the state vector ψ is written $|\psi\rangle$ and one can conveniently designate an eigenstate whose eigenvalue is, say, 3 by $|3\rangle$. Then we could restate Born's rule above to say that if the state of the system is

$|\psi\rangle = c_1|\lambda_1\rangle + c_2|\lambda_2\rangle + \ldots$, then $\mathrm{Pr}ob_{|\psi\rangle}(A = \lambda_n) = |c_n|^2$.

Dirac writes $\langle A \rangle_\psi$ as $\langle \psi | A | \psi \rangle$. Corollary 2 says that $\Pr ob_{|\lambda\rangle}(A = \lambda) = 1$.

Eigenvalues and eigenstates. If $A|\varphi\rangle = \lambda|\varphi\rangle$ then $|\varphi\rangle$ is an eigenstate of *A* with eigenvalue λ. Here *A* is an operator, $|\varphi\rangle$ is a vector in the state space, $A|\varphi\rangle$ is the result of the operator *A* applied to the vector $|\varphi\rangle$ and λ is a number (in general a complex number -- in special cases it can be real.)

Inner product. There is an "inner" product of any two vectors ψ and φ that we write in Dirac notation as $\langle \psi | \varphi \rangle$. This designates a number, generally complex. Order counts so that in general $\langle \psi | \varphi \rangle \neq \langle \varphi | \psi \rangle$. In the special cases where equality holds, $\langle \psi | \varphi \rangle$ is a real number. In the superposition for $|\psi\rangle$ in the above statement of Born's Rule $c_n = \langle \psi | \lambda_n \rangle$. Also the expected value $\langle \psi | A | \psi \rangle$ is the inner product of $|\psi\rangle$ with $A|\psi\rangle$ (where $A|\psi\rangle$ is the result of the operator *A* applied to the vector $|\psi\rangle$).

Measurement and Collapse. After a measurement of the observable *A* on a system initially in state

$$|\psi\rangle = c_1|\lambda_1\rangle + c_2|\lambda_2\rangle + \ldots$$

(where the $|\lambda_n\rangle$ are the *A*-eigenstates) the post-measurement state becomes one of the $|\lambda_n\rangle$ with probability $|c_n|^2$.

This transition from $|\psi\rangle$ to $|\lambda_n\rangle$ is usually called the "collapse" of the state vector (or “wave packet”) or sometimes "the projection postulate". The collapse does not satisfy the Schrödinger evolution described in the chart above, (5), giving rise to one aspect of the so-called "measurement problem".

Now Go to Interacting Systems.