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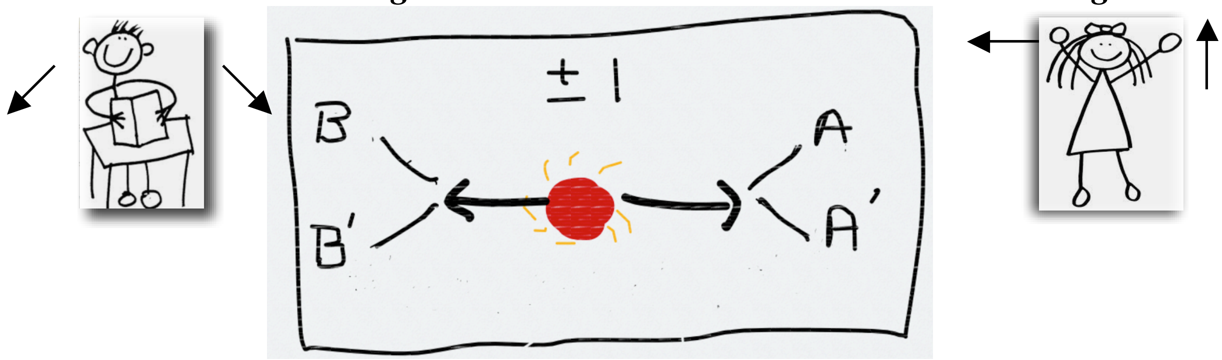
THE BELL THEOREM

The Bell theorem is a demonstration that a plausible way of picturing (or modeling) how the results of measurements occur, when made precise, leads to definite relations between the statistics in different measurements (the "Bell inequalities"), relations that are violated for a certain class of quantum mechanical experiments.

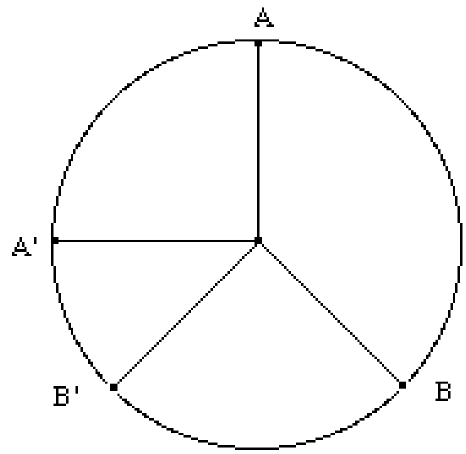
In the experiments an atomic source steadily decays and emits pairs of spin-1/2 particles (e.g., electrons) in the singlet state (so, the total spin is 0, and it is conserved). After emission, the electrons in a pair separate and move in opposite directions to distinct wings of the experimental apparatus for observation. In the wings there are instruments that can be set to measure the spin component of an electron along one of two directions in the plane perpendicular to the electron's line of motion.

Bernard's Wing ***Amalie's Wing***



RUNS: A_B, A_B', A'_B, A'_B'

Thus the experiment involves the measurement of four variables either A or A' (for one particle in a pair) and either B or B' (for the other particle in the same pair), corresponding to the four possible orientations for the measurement of spin. A particular spin measurement has two possible outcomes: the electron is either spinning clockwise ("up") or counterclockwise ("down") in the measured direction. We will record the "up" result with a "+1" and the "down" with a "-1". Under these conventions each of the four variables takes either +1 or -1 as a value. In four separate runs, on paired particles, we measure orientations A with B, then A with B' then A' with B, and finally A' with B'. The best tested experimental geometry corresponds to the situation where the relative angle between directions in the first three runs (A/B, A/B', and A'/B) is $\mathbf{135^{o}}$ and the relative angle between the orientations in the last run (A'/B') is $\mathbf{45^{o}}$, as in the diagram below.



In a number of experiments of this type (more accurately, in experiments involving photon polarization, which is formally similar to electron spin) the statistics predicted by the quantum theory have been very well-confirmed. (See Weihs *et al* 1998.) The statistics are these (Clauser and Horne 1974).

$$P(A) = P(B) = P(A') = P(B') = 0.5 \tag{1}$$

$$P(AB) = P(AB') = P(A'B) = \frac{1}{2}\sin^2\left(\frac{135^o}{2}\right) = 0.4268 \qquad \text{and} \tag{2i}$$

$$P(A'B') = \frac{1}{2}\sin^2\left(\frac{45^o}{2}\right) = 0.0732 \tag{2ii}$$

(We write P(A) for Prob(A=+1) and P(AB) for Prob(A=+1 & B=+1), and similarly for the other variables.)

Suppose that there were factors ("hidden variables") that determined the experimental outcomes, or more generally, their probability. Let "λ" range over the factors that determine the probability of a measurement outcome in each individual case. Thus if an emitted pair is characterized by a particular factor λ and we measure A (or A') on one particle in the pair then λ (or its relevant component) determines the probability that the result of the A-measurement is +1; similarly λ would determine the probability for the outcome to be +1 if we measure B (or B') on the other particle in the pair. **This understanding represents the probability for the (+1) measurement outcomes by "response" functions p(A, λ), p(B, λ), p(A', λ) and p(B', λ) taking values between 0 and 1, depending on the determining factor λ (LOCALITY)** .We will show that under certain plausible assumptions, such a representation is inconsistent with the data in (1) and (2).

For that purpose, notice that if numbers p, q and r all lie between 0 and 1, then

$$qr = pqr + (1-p)qr \leq pr + (1-p)q = pr - pq + q,$$

since multiplying by numbers between 0 and 1 may only decrease the value of the original. Thus the following inequality holds

$$qr \le pr - pq + q .$$

Because averaging over the values of variables preserves sums, differences and order, it follows that

$$\langle qr\rangle \le \langle pr\rangle - \langle pq\rangle + \langle q\rangle \qquad \text{(Basic Inequality)}$$

if p, q, and r are variables taking values between 0 and 1, and $\langle\cdot\rangle$ denotes the average (or expected) value of the enclosed variable(s).

To connect this with the correlation experiment, we suppose that the observed probability for a measurement outcome in a run is simply the average (over all the hidden factors) of the probabilities for individual outcomes; that is, we assume

$$P(A) = \langle p(A, \lambda) \rangle$$

(and similarly for P(B), P(B'), and P(A'). We also assume that the outcomes at λ are uncorrelated, so the probability $p(AB, \lambda)$ at a factor λ for the pair of outcomes $A = +1$ and $B = +1$ is the product of the probabilities at λ for each outcome separately; that is

$$\mathbf{p(AB, \lambda) = p(A, \lambda)\cdot p(B, \lambda).} \qquad \textbf{(FACTORIZABILITY)}$$

Then, as in the case of the individual outcomes, the probability for pairs is obtained by averaging over all the hidden factors; i.e.,

$$P(AB) = \langle p(AB, \lambda)\rangle = \langle p(A, \lambda)\cdot p(B, \lambda)\rangle .$$

(and similarly for the other pairs AB', A'B and A'B'.)

If in the basic inequality we now set $q = p(B, \lambda)$, $r = p(B', \lambda)$ and $p = p(A', \lambda)$ then, when we substitute the overall probabilities for the averages as above, we find that

$$\langle p(B, \lambda)\cdot p(B', \lambda)\rangle \le P(A'B') - P(A'B) + P(B).$$

If we perform the same substitutions for the averages, but this time setting $q = p(A, \lambda)$, $r = p(B', \lambda)$, and $p = p(B, \lambda)$ in the basic inequality, then (after transposing) we have that

$$P(AB') + P(AB) - P(A) \le \langle p(B, \lambda)\cdot p(B', \lambda)\rangle.$$

Combining these two inequalities yields

$$\mathbf{P(AB) + P(AB') + P(A'B) - P(A'B') \le P(A) + P(B)}. \qquad \textbf{(Bell Inequality)}$$

From (2) the left hand side equals 1.207 and from (1) the right hand side equals 1, in violation of the inequality. Thus the supposition that there are factors determining the probabilities for measurement outcomes, as above, leads to the Bell inequality, which is inconsistent with the experimental data in (1) and (2).

In addition to **factorizability**, which blocks any correlation between the probable outcomes at λ, the preceding demonstration involves **"locality"** assumptions about the measurement procedure. In representing the probabilities for measurement outcomes as functions of certain determining factors λ, we suppose that the probability for an outcome of a measurement in one wing, say A, is not affected by which of the variables (B or B') is measured in the other wing. If there were correlations between the measurement performed in one wing and the probability for an outcome obtained in the other wing we should have to represent the probability that A=+1 at λ not by $p(A, \lambda)$ but by $p(A, \lambda, B)$ or $p(A, \lambda, B')$. In that case, however, the preceding argument would not go through. For similar reasons we have to assume that the factor λ that determines the outcome-probabilities does not depend on the particular measurements being carried out in the wings ("**Measurement Independence**").

In the quantum mechanical experiment described above we can arrange things so as to be reasonably certain that the spin measurements of the particles in an emitted pair are spacelike-separated; i.e., that the systems are so far apart and the times of separate measurements on a pair are so close that no signals (or "influences") with speeds less than (or equal to) that of light can pass between them (Aspect *et al* 1982 and Weihs *et al* 1998). To insure measurement independence (sometimes called "freedom of choice") we also want the emission of a pair with a given λ to be spacelike-separated from the operations that set the measurements to be performed. This too has been accomplished (Scheid *et al* 2010).

APPENDIX

There is another relation implied by the preceding locality and factorizability assumptions, one that bounds the quantity [P(AB) + P(AB') + P(A'B) - P(A'B')] from below. We can derive it using an argument similar to the one above. So suppose, as above, that the numbers p, q and r all lie between 0 and 1. Then

$$\begin{aligned}(1-p)(1-q)(1-r) &= (1-p)(1-r) - q(1-p)(1-r)\\ &= 1 - p - r + pr - q(1-p) + q(1-p)r\\ &\leq 1 - p - r + pr - q + pq + qr.\end{aligned}$$

So, since $0 \leq (1-p)(1-q)(1-r)$,

$$p + q + r \leq 1 + pq + pr + qr.$$

If we now set $q = p(B, \lambda)$, $r = p(B', \lambda)$ and $p = p(A, \lambda)$, and take the average over all the factors λ, then we have that

$$P(A) + P(B) + P(B') \leq 1 + P(AB) + P(AB') + \langle p(B, \lambda)\cdot p(B', \lambda)\rangle$$

i.e., that

$$P(A) + P(B) + P(B') - 1- P(AB) - P(AB') \leq \langle p(B, \lambda)\cdot p(B', \lambda)\rangle.$$

Also,

$$qr = pqr + (1-p)qr \leq pq + (1-p)r = pq - pr + r,$$

so

$$qr \leq pq - pr + r.$$

Putting $q = p(B, \lambda)$, $r = p(B', \lambda)$ and $p = p(A', \lambda)$, and averaging, produces

$$\langle p(B, \lambda) \cdot p(B', \lambda) \rangle \leq P(A'B) - P(A'B') + P(B').$$

Putting together the inequalities involving $\langle p(B, \lambda) \cdot p(B', \lambda) \rangle$ yields

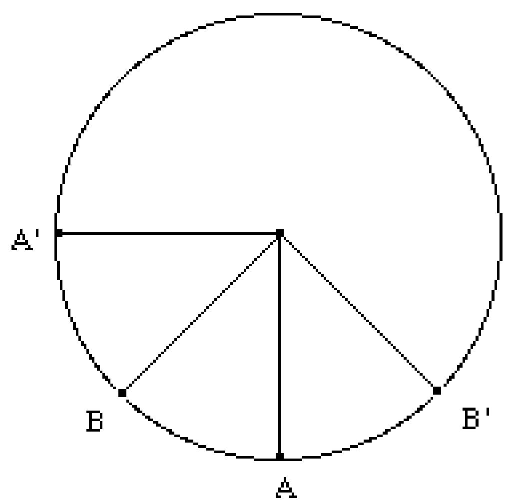
$$P(A) + P(B) + P(B') - 1 - P(AB) - P(AB') \leq P(A'B) - P(A'B') + P(B'),$$

or

$$\mathbf{P(A) + P(B) - 1 \leq P(AB) + P(AB') + P(A'B) - P(A'B').} \qquad \text{(Bell Inequality)}$$

As with the first Bell inequality this one is violated in the spin-1/2 experiments as well. Consider the situation where the relative angle between directions in the first three runs (A/B, A/B', and A'/B) is 45^{o} and the relative angle between the orientations in the last run (A'/B') is 135^{o}, as in the diagram below.

$$P(AB) = P(AB') = P(A'B) = \frac{1}{2} \sin^2\left(\frac{45^{o}}{2}\right) = 0.0732$$



$$P(A'B') = \frac{1}{2} \sin^2\left(\frac{135^{o}}{2}\right) = 0.4268.$$

Here, again, all the single probabilities are $\frac{1}{2}$. The preceding Bell inequality then implies that $0 \leq -0.207$! If we combine the two Bell inequalities, we obtain

$$\mathbf{P(A) + P(B) - 1 \leq P(AB) + P(AB') + P(A'B) - P(A'B') \leq P(A) + P(B).}$$

In the spin-1/2 case the single probabilities are all $\frac{1}{2}$ and this reduces to

$$\mathbf{0 \leq P(AB) + P(AB') + P(A'B) - P(A'B') \leq 1.}$$

There are three more Bell inequalities, obtained from the above by first interchanging A with A', then B with B', and finally both A with A' and B with B' together.

The system of all these inequalities taken together constitutes the necessary and sufficient conditions that a correlation experiment involving the four distinct orientations A, A', B and B' has a statistical model that determines the probabilities for measurement outcomes in accord with the twin requirements of locality and factorizability (Fine 1982). The failure of the quantum statistics to satisfy these inequalities shows that it is not possible to model the quantum probabilities in this way. Of course there may be other sorts of models. Moreover, when inefficiencies of actual experiments are taken into account some models of the sort discussed here also become possible.

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