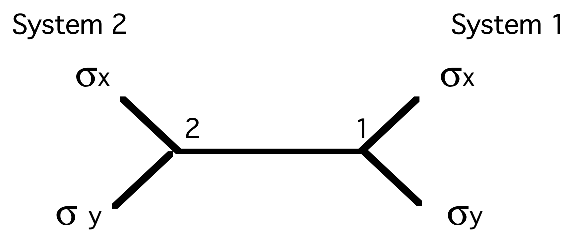
THE KOCHEN-SPECKER/BELL/ FINE-TELLER THEOREM: a la PERES

We consider an EPR/Bohm spin-1/2 experiment, schematized below, where the σs are the spin-component operators (or, ambiguously, the spin observables) defined on the two subsystems (System 1 & System 2).



The state ψ of the two-particle composite system is the "singlet state," which can be written as

$$\psi = \frac{1}{\sqrt{2}}\,[\phi_1^+(\alpha) \otimes \phi_2^-(\alpha)] - \frac{1}{\sqrt{2}}\,[\phi_1^-(\alpha) \otimes \phi_2^+(\alpha)]$$

where α may be any direction (including x, y, or z) and where

$$\sigma_\alpha^1 = \text{the spin component in direction } \alpha \text{ on System 1} = \begin{cases} +1 & \text{in state } \phi_1^+(\alpha) \\ -1 & \text{in state } \phi_1^-(\alpha) \end{cases}$$ and

$$\sigma_\alpha^2 = \text{the spin component in direction } \alpha \text{ on System 2} = \begin{cases} +1 & \text{in state } \phi_2^+(\alpha) \\ -1 & \text{in state } \phi_2^-(\alpha) \end{cases}.$$

In state ψ the total spin in any direction is zero; i.e., if the spin is found to be "up" (+1) in direction α on one system it will be "down" (−1) in direction α on the other system, and *vice versa* . (Put otherwise, the product of the spin components on the separate systems in the same direction is always −1.) Suppose we make a measurement of σ_x^1 on System 1 and (simultaneously) of σ_y^2 on System 2, finding, respectively, values **a** and **b**. We can then cross-infer the values for σ_x^2 and for σ_y^1, to get the results as follows.

Observables: σ_x^1 , σ_y^2 , σ_x^2 , σ_y^1

Values: **a** , **b** , **-a** , **-b**

Since the only possible values for **a** and **b** are ±1, if we multiply them together the product (**ab**) will also be ±1, and its square $(\mathbf{ab})^2 = 1$. So we have that

$$\begin{aligned}\text{val}\,[(\sigma_x^1 \cdot \sigma_y^2)(\sigma_y^1 \cdot \sigma_x^2)] &= \text{val}\,(\sigma_x^1 \cdot \sigma_y^2) \cdot \text{val}\,(\sigma_y^1 \cdot \sigma_x^2) \\ &= \text{val}\,(\sigma_x^1) \cdot \text{val}\,(\sigma_y^2) \cdot \text{val}\,(\sigma_y^1) \cdot \text{val}\,(\sigma_x^2) \\ &= \mathbf{a} \cdot \mathbf{b} \cdot (-\mathbf{b}) \cdot (-\mathbf{a}) \\ &= (\mathbf{ab})^2.\end{aligned}$$

Hence,

$$\text{val}\,[(\sigma_x^1 \cdot \sigma_y^2)(\sigma_y^1 \cdot \sigma_x^2)] = 1. \qquad (*)$$

However, by using standard relationships between the various spin operators we can calculate the value of the product $[(\sigma_x^1 \cdot \sigma_y^2)(\sigma_y^1 \cdot \sigma_x^2)]$ in a second way. First of all, the spin operators in orthogonal directions (like "x" and "y" above) on a single system (say, System 2) anti-commute; that is,

$$(\sigma_x^2 \cdot \sigma_y^2) = -(\sigma_y^2 \cdot \sigma_x^2).$$

Also, spin in the z-direction is related to spin in the x- and y-directions by the equations

$$\sigma_z^1 = \mathbf{i}\,(\sigma_x^1 \cdot \sigma_y^1)$$

$$\sigma_z^2 = \mathbf{i}\,(\sigma_x^2 \cdot \sigma_y^2)$$

where $\mathbf{i}^2 = -1$. Using $(\sigma_x^2 \cdot \sigma_y^2) = -(\sigma_y^2 \cdot \sigma_x^2)$ we can rewrite the expression for σ_z^2,

$$\sigma_z^2 = -\mathbf{i}\,(\sigma_y^2 \cdot \sigma_x^2).$$

Using the fact that $(\sigma_y^1 \cdot \sigma_y^2) = (\sigma_y^2 \cdot \sigma_y^1)$ we can readily calculate $(\sigma_z^1 \cdot \sigma_z^2)$ from the above relations as follows.

$$(\sigma_z^1 \cdot \sigma_z^2) = \mathbf{i}\,(\sigma_x^1 \cdot \sigma_y^1) \cdot (-\mathbf{i})\,(\sigma_y^2 \cdot \sigma_x^2) = -(\mathbf{i}^2)\,(\sigma_x^1 \cdot \sigma_y^1)(\sigma_y^2 \cdot \sigma_x^2) = (\sigma_x^1 \cdot \sigma_y^2)(\sigma_y^1 \cdot \sigma_x^2).$$

Thus, reading from far left to far right,

$$\boxed{\text{val}\,(\sigma_z^1 \cdot \sigma_z^2) = \text{val}\,[(\sigma_x^1 \cdot \sigma_y^2)(\sigma_y^1 \cdot \sigma_x^2)].}$$

According to (*), the RHS = +1.

Now we know that for direction z, like any other, the spin values on the two systems are opposite to one another; i.e., that

$$\text{val}\,(\sigma_z^1 \cdot \sigma_z^2) = \text{val}\,(\sigma_z^1) \cdot \text{val}\,(\sigma_z^2) = -1. \qquad (**)$$

(Indeed, the singlet state ψ is an eigenstate of ($\sigma_z^1 \cdot \sigma_z^2$) with eigenvalue −1, so the only possible value for ($\sigma_z^1 \cdot \sigma_z^2$) is −1.)

So from (**) the LHS of the boxed equation = −1. Since $-1 \neq 1$ (!), it follows that we cannot consistently assign values as above.

An examination of the calculations shows that the principles we used to assign values are just these:

1) The only possible values for an observable A of a system in a state ψ are the eigenvalues of A that have non-zero probability in ψ. (EIGENVALUE PRINCIPLE)

2) If A and B commute, then val (A·B) = val (A)·val (B). (PRODUCT RULE)

Thus we have established the following "no-go" ("*Dass geht nicht.*") theorem.
<u>THEOREM</u>:

THERE IS NO ASSIGNMENT OF EXACT VALUES TO THE QUANTUM OBSERVABLES THAT SATISFIES THE EIGENVALUE PRINCIPLE AND THE PRODUCT RULE.

REFERENCES

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