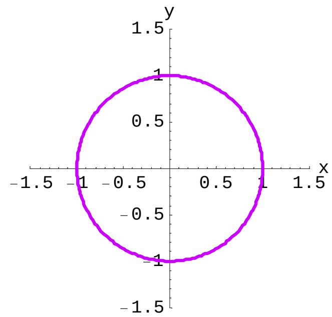
Lecture 13 – Appendix B: Some sample problems from Boas

Here are some solutions to the sample problems assigned for Chapter 6.8 to 6.11.

§6.8: 2

Solution: We want to practice doing closed line integrals of the form $\oint (x+2y)\,dx - 2x\,dy \equiv \oint \vec{F}\cdot d\vec{r}$ clockwise along a set of curves. Let us first check the curl of the vector in the integrand. We have

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+2y & -2x & 0 \end{vmatrix} = \hat{z}\left(-2-2\right) = -4\hat{z} \neq 0.$$



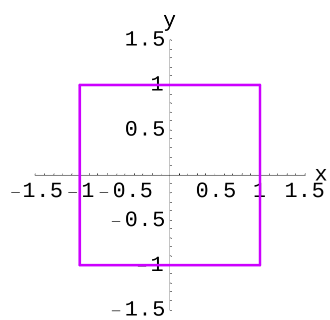
Thus we expect nonzero (path-dependent) closed path integrals

a) On a circle of radius 1: here we can use cylindrical coordinates (see example 2) and find

$$\oint (x+2y)\,dx - 2x\,dy = \int_0^{2\pi} d\theta\left[\left(\cos\theta + 2\sin\theta\right)\left(-\sin\theta\right) - 2\cos\theta\cos\theta\right]$$

$$= \int_0^{2\pi} d\theta\left[-\sin\theta\cos\theta - 2\left(\sin^2\theta + \cos^2\theta\right)\right] = -\int_0^{2\pi} d\theta\left[\frac{1}{2}\sin 2\theta + 2\right] = -4\pi.$$

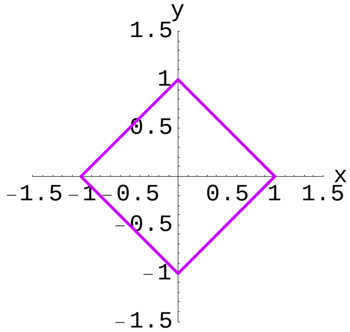
b) On the square of side 2 with sides aligned with axes and starting at (1,1): writing out the contributions from the 4 sides (only *x or y* varies on each side), we find



$$\oint (x+2y)\,dx - 2x\,dy = \int_1^{-1} dx\,(x+2) + \int_1^{-1} dy\,(2) + \int_{-1}^{1} dx\,(x-2) + \int_1^{-1} dy\,(-2)$$

$$= \left[\frac{x^2}{2} + 2x\right]_1^{-1} + \left[2y\right]_1^{-1} + \left[\frac{x^2}{2} - 2x\right]_{-1}^{1} + \left[-2y\right]_{-1}^{1} = -4-4-4-4 = -16.$$

c) On the square of side $\sqrt{2}$, rotated by 45° with respect to the axis and starting at (0,1). With constant slope on each side we can express both dx and dy in terms of a single variable, say dt, $d\vec{r} = \pm\hat{x}dt \pm \hat{y}dt$. Thus we have



$$\oint (x+2y)\,dx - 2x\,dy = \int_0^1 dt\left[-(x+2y)+2x\right]_{(x=-t,\,y=1-t)} + \int_0^1 dt\left[(x+2y)+2x\right]_{(x=t-1,\,y=-t)}$$

$$+\int_0^1 dt\left[(x+2y)-2x\right]_{(x=t,\,y=t-1)} + \int_0^1 dt\left[-(x+2y)-2x\right]_{(x=1-t,\,y=t)}$$

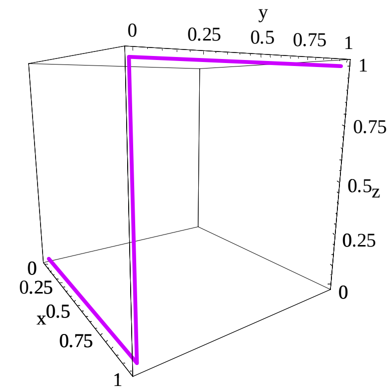
$$= \int_0^1 dt\left[(t-2)+(t-3)+(t-2)+(t-3)\right] = \int_0^1 dt\left[4t-10\right] = \left[2t^2-10t\right]_0^1 = -8.$$

§6.8: 4

<u>Solution</u>: Now we want to perform a(n open) line integral, $\int_C y^2dx + 2xdy + dz \equiv \int_C \vec{F}\cdot d\vec{r}$, along 2 different paths. Again we check the curl to see if there is path dependence expected. We find a non-zero curl and expect path dependence

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2x & 1 \end{vmatrix} = \hat{x}(0) + \hat{y}(0) + \hat{z}(2-2y) = 2\hat{z}(1-y) \neq 0.$$

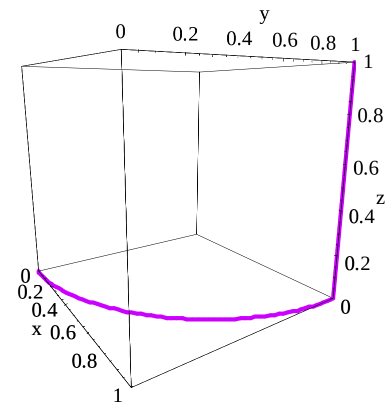
a) First we consider a path composed of straight line segments parallel to the axes. Thus we have



$$\int_C y^2 dx + 2xdy + dz = \int_0^1 dx\left[y^2\right]_{y=0} + \int_0^1 dz + \int_0^1 dy\left[2x\right]_{x=z=1}$$

$$= 0 + 1 + 2 = 3.$$

b) Now along a path composed of an arc of a circle in the x-y plane ($x^2 + y^2 - 2y = 0 \Rightarrow (x-0)^2 + (y-1)^2 = 1$) and then parallel to the z-axis. Using cylindrical coordinates on the former ($x = \sin\theta,\ y = 1-\cos\theta$) we have



$$\int_C y^2 dx + 2xdy + dz = \int_0^{\pi/2} d\theta\left[\cos\theta(1-\cos\theta)^2 + \sin\theta(2\sin\theta)\right] + \int_0^1 dz$$

$$= \int_0^{\pi/2} d\theta\left[\cos\theta\left(1+\cos^2\theta\right) - 2\left(\cos^2\theta - \sin^2\theta\right)\right] + 1$$

$$= \int_0^{\pi/2} d\theta\left[\cos\theta\left(2-\sin^2\theta\right) - 2\left(\cos 2\theta\right)\right] + 1$$

$$= \left[2\sin\theta - \frac{\sin^3\theta}{3} - \sin 2\theta\right]_0^{\pi/2} + 1 = 2 - \frac{1}{3} + 1 = \frac{8}{3}.$$

§6.8: 8

Solution: We want to verify a conservative force and find the potential for $\vec{F} = \hat{x} - z\hat{y} - y\hat{z}$. We take the curl and find that it vanishes,

$$\vec{\nabla}\times\vec{F}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 1 & -z & -y\end{vmatrix}=\hat{x}\left(-1+1\right)+\hat{y}\left(0\right)+\hat{z}\left(0\right)=0.$$

Next we want to find a potential ϕ such that (*i.e.*, we perform some trivial integrals)

$$\vec{F}=-\vec{\nabla}\phi\Rightarrow\begin{cases}-\frac{\partial\phi}{\partial x}=1\\ -\frac{\partial\phi}{\partial y}=-z\\ -\frac{\partial\phi}{\partial z}=-y\end{cases}\Rightarrow\phi=-x+yz+\text{constant}.$$

§6.8: 11

<u>Solution</u>: Now the same game as in the previous exercise except that now $\vec{F}=y\sin 2x\hat{x}+\sin^2 x\hat{y}$. So the curl again vanishes,

$$\vec{\nabla}\times\vec{F}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y\sin 2x & \sin^2 x & 0\end{vmatrix}$$
$$=\hat{x}\left(0\right)+\hat{y}\left(0\right)+\hat{z}\left(2\cos x\sin x-\sin 2x\right)=0.$$

We find the potential from

$$\vec{F} = -\vec{\nabla}\phi \Rightarrow \begin{cases} -\dfrac{\partial \phi}{\partial x} = y\sin 2x \\ -\dfrac{\partial \phi}{\partial y} = \sin^2 x \\ -\dfrac{\partial \phi}{\partial z} = 0 \end{cases} \Rightarrow \phi = -y\sin^2 x + \text{constant.}$$

§6.8: 14

<u>Solution</u>: Finally for $\vec{F} = \hat{x}\, y\Big/\sqrt{1-x^2y^2} + \hat{y}\, x\Big/\sqrt{1-x^2y^2}$ we find

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y\Big/\sqrt{1-x^2y^2} & x\Big/\sqrt{1-x^2y^2} & 0 \end{vmatrix}$$

$$= \hat{x}(0) + \hat{y}(0) + \hat{z}\left(\frac{1}{\sqrt{1-x^2y^2}} - \frac{1}{\sqrt{1-x^2y^2}} + \frac{x^2y^2 - x^2y^2}{\left(\sqrt{1-x^2y^2}\right)^3}\right) = 0,$$

and (this requires recognizing the derivatives of the arcsine function)

$$\vec{F} = -\vec{\nabla}\phi \Rightarrow \begin{cases} -\dfrac{\partial \phi}{\partial x} = y\Big/\sqrt{1-x^2y^2} \\ -\dfrac{\partial \phi}{\partial y} = x\Big/\sqrt{1-x^2y^2} \\ -\dfrac{\partial \phi}{\partial z} = 0 \end{cases} \Rightarrow \phi = -\sin^{-1} xy + \text{constant.}$$

§6.9: 4

Solution: We want to practice using the 2D Green's theorem to perform the indicated contour integral. We have

$$\begin{aligned} I &= \int_{C=ADB} e^x \cos y dx - e^x \sin y dy \\ &= \oint_{C'=ADBA} e^x \cos y dx - e^x \sin y dy - \int_{C''=BA} e^x \cos y dx - e^x \sin y dy \\ &= \oint_{C'=ADBA} P\big|_{P=e^x \cos y} dx + Q\big|_{Q=-e^x \sin y} dy - \int_{-\ln 2}^{\ln 2} e^x \cos y\big|_{y=0} dx \\ &= \oint_{C'=ADBA} P\big|_{P=e^x \cos y} dx + Q\big|_{Q=-e^x \sin y} dy - e^x\Big|_{-\ln 2}^{\ln 2} \\ &= \oint_{C'=ADBA} Pdx + Qdy - \frac{3}{2}. \end{aligned}$$

Thus applying Green's theorem we find

$$\begin{aligned} I &= \iint_{\text{Area}=ADBA} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy - \frac{3}{2} \\ &= \iint_{\text{Area}=ADBA} \left(-e^x \sin y - \left(-e^x \sin y \right) \right) dxdy - \frac{3}{2} = 0 - \frac{3}{2} = -\frac{3}{2}. \end{aligned}$$

§6.9: 7

Solution: We want to use the result in exercise 6.9:6 to calculate the area of an ellipse defined by $x = a\cos\theta$, $y = b\sin\theta$, with $0 \le \theta < 2\pi$. The result in 6.9:6 is based on the 2-D Green's theorem with the special choices $P = -y/2$, $Q = x/2$. With our choices to parameterize the ellipse we have $dx = -a\sin\theta d\theta$ and $dy = b\cos\theta d\theta$ to yield

$$A = \frac{1}{2}\int_0^{2\pi} d\theta \left(ab\cos^2\theta + ab\sin^2\theta \right) = \pi ab.$$

§6.9: 10

Solution: For the path in the (x,y) plane defined by the 4 points (3,1), (5,1), (5,3), (3,3), we want to evaluate the line integral

$$\oint \left(2y\,dx - 3x\,dy \right).$$

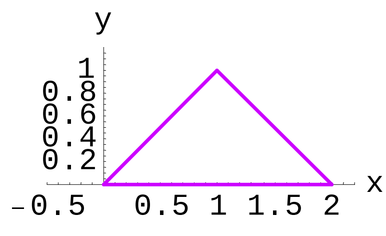
This expression suggests that we define $P = 2y$ and $Q = -3x$ and consider the expression in the 2D Green's theorem, $\partial Q/\partial x - \partial P/\partial y = -3 - 2 = -5$. Since this is a constant, Green's theorem tells us that we need only the area of the square defined by the 4 points (with sides of length 2), which is just 4. Thus the easy approach, use Green, is

$$\oint \left(2y\,dx - 3x\,dy \right) = -5 \iint_{\text{square}} dx\,dy = -20.$$

On the other hand proceeding directly (and laboriously) we find

$$\begin{aligned} &\oint \left(2y\,dx - 3x\,dy \right) \\ &= \int_3^5 dx \left(2y \right)_{y=1} + \int_1^3 dy \left(-3x \right)_{x=5} + \int_5^3 dx \left(2y \right)_{y=3} + \int_3^1 dy \left(-3x \right)_{x=3} \\ &= 2\int_3^5 dx - 15\int_1^3 dy - 6\int_3^5 dx + 9\int_1^3 dy = 4 - 30 - 12 + 18 = -20. \end{aligned}$$

§6.9: 11



Solution: Here is one more contour integral over the indicated triangle in the clockwise direction. Using Green's theorem we have

$$I = \oint_{C=\text{triangle}} \left(x\sin x - y\right)dx + \left(x - y^2\right)dy$$

$$= \iint_{\text{triangle}} \left(\frac{\partial\left(x-y^2\right)}{\partial x} - \frac{\partial\left(x\sin x - y\right)}{\partial y} \right) dxdy$$

$$= \iint_{\text{triangle}} \left(1-(-1)\right)dxdy = 2\text{area} = 2\frac{1\cdot 2}{2} = 2.$$

§6.10: 4

<u>Solution</u>: Now we want to practice using the divergence theorem to relate surface and volume integrals. In this case the vector field is $\vec{V} = x\cos^2 y\hat{x} + xz\hat{y} + z\sin^2 y\hat{z}$ and the boundary surface is a sphere of radius 3. Thus we find

$$\oiint_{r=3} \vec{V}\cdot d\vec{\sigma} = \iiint_{r=3} \vec{\nabla}\cdot\vec{V}d\tau = \iiint_{r=3} \left(\cos^2 y + \sin^2 y\right)d\tau = \iiint_{r=3} d\tau$$

$$= \left.\frac{4\pi r^3}{3}\right|_{r=3} = 36\pi.$$

This approach is clearly simpler than during the surface integral directly!

§6.10: 7

<u>Solution</u>: Next we consider a similar problem where the surface is a cone of height 3 and a base of radius 4, and the vector field is $\vec{V} = \vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$. Thus we find (again this the simpler approach)

$$\oiint_{\text{cone}} \vec{r}\cdot d\vec{\sigma} = \iiint_{\text{cone}} \vec{\nabla}\cdot\vec{r}d\tau = \iiint_{\text{cone}} (3)\,d\tau = 3\iiint_{\text{cone}} d\tau$$

$$= 3\left.\frac{\pi r^2 h}{3}\right|_{h=3,r=4} = 48\pi.$$

§6.10: 9

<u>Solution</u>: Now we want to use the divergence theorem to evaluate the surface integral of $\vec{F} = x\hat{x} + y\hat{y} = \vec{\rho}$ on the surface defined by $z = 4 - x^2 - y^2 = 4 - \rho^2$,

$$\iint_{z=4-x^2-y^2=4-\rho^2} d\vec{\sigma} \cdot \vec{F}.$$

Note the explicit forms for the surface and the function suggest the use of cylindrical coordinates. First check the divergence to see if is usefully simple. Since $\vec{\nabla} \cdot \vec{F} = \hat{\rho} \cdot \left(\partial\vec{\rho} / \partial\rho \right) = 2$ (recall the previous appendix or proceed in rectangular coordinates), *i.e.*, a constant, the divergence theorem will be useful. We can close the surface by adding the disk in the *x*,*y* plane where the outward normal is just $\hat{n} = -\hat{z}$ and $\vec{F} \cdot \hat{n}\Big|_{\text{disk}} = -\vec{F} \cdot \hat{z} = 0$. This the desired surface integral is just the volume integral of the divergence,

$$\iint_{z=4-x^2-y^2=4-\rho^2} d\vec{\sigma} \cdot \vec{F} = \iiint_{0<z<4-\rho^2} d\text{vol}\left(\vec{\nabla} \cdot \vec{F} \right) = 2\int_0^2 \rho d\rho \int_0^{2\pi} d\phi \int_0^{4-\rho^2} dz$$

$$= 4\pi \int_0^2 \rho d\rho \left(4 - \rho^2 \right) = 4\pi \left[2\rho^2 - \frac{\rho^4}{4} \right]_0^2 = 16\pi.$$

It is informative, if challenging, to also try evaluating the original surface integral directly. We know that the normal to the surface is given by the (normalized) gradient of the function defining the surface,

$$d\vec{\sigma} = d\sigma\hat{n},\, \hat{n} = \frac{\vec{\nabla}\left(z + \rho^2 \right)}{\left| \vec{\nabla}\left(z + \rho^2 \right) \right|} = \frac{2\rho\hat{\rho} + \hat{z}}{\sqrt{4\rho^2 + 1}}$$

$$\vec{F} \cdot d\vec{\sigma} = \frac{2\rho^2}{\sqrt{4\rho^2 + 1}} d\sigma.$$

So the remaining challenge is to usefully express the differential area $d\sigma$. To this end it is helpful to read Section 5.5 in Boas where it is pointed out that, for the area of a surface above the *x,y* plane, we can express the local differential area as $d\sigma = \sec\gamma\, dxdy = \sec\gamma \rho d\rho d\phi$, where the angle γ is the angle between the local normal to the surface (as worked out above) and the *z* direction, $\sec\gamma = 1/\left|\hat{n}\cdot\hat{z}\right| = \sqrt{4\rho^2+1}$. This factor, which is ≥ 1, just accounts for how much larger the area of the true surface is compared to its projection onto the *x,y* plane. Thus we have in our case

$$\iint_{z=4-x^2-y^2=4-\rho^2} d\vec{\sigma}\cdot\vec{F} = \int_0^{2\pi} d\phi \int_0^2 \rho d\rho \left(\frac{2\rho^2}{\sqrt{4\rho^2+1}}\right)\left(\sqrt{4\rho^2+1}\right)$$
$$= 4\pi\int_0^2 \rho^3 d\rho = \pi\,\rho^4\Big|_0^2 = 16\pi.$$

Again we see that the actual integration is much simpler using the divergence theorem.

§6.10: 12

<u>Solution</u>: This exercise concerns an electrostatics problem with concentric charged cylindrical conductors, with radii R_1 and R_2, and with *k* coulombs per meter and –*k* coulombs per meter, respectively. We can use Gauss's law applied (per unit length) to find the electric field as function of the radius and then, by integration, the corresponding potential. Clearly we want to use cylindrical coordinates, ρ, ϕ, z in my notation. We begin by using the translational symmetry in the *z* direction (the axis of the cylinders and the cylinders are "very long") to argue that the electric field has no variation with the *z* coordinate. Likewise the rotational symmetry about the *z* axis means there is no dependence on the azimuthal angle ϕ. Since we are assuming that this is a static situation with no moving charges, there can be no nonzero components of the electric field in the surface of the conductors (otherwise charges would move). Hence, by general arguments, we have only a ρ component and it varies only with ρ, $\vec{E}(\vec{r}) = E(\rho)\hat{\rho}$. So now we apply Gauss for the three cases: $\rho < R_1$, $R_1 < \rho < R_2$, and $\rho > R_2$. In the first and last cases there is no net charge per unit length and thus the electric field must vanish,

$$\rho < R_1 : \iint_{\rho<R_1} \vec{E}\cdot d\vec{\sigma} = E(\rho)\rho\Big|_{\rho<R_1} \iint d\phi dz = E(\rho)2\pi\rho\Big|_{\rho<R_1} \int dz$$

$$= \iiint_{\rho<R_1} d\text{vol}\,\frac{q}{\varepsilon_0} = \int_0^{\rho<R_1} \bar{\rho}\,d\bar{\rho}\int_0^{2\pi} d\phi \int dz\,\frac{q}{\varepsilon_0} = 0\times\int dz \Rightarrow E(\rho<R_1)=0,$$

$$\rho > R_2 : E(\rho)\rho\Big|_{\rho>R_2} \iint d\phi dz = E(\rho)2\pi\rho\Big|_{\rho>R_2} \int dz$$

$$= \iiint_{\rho<R_1} d\text{vol}\,\frac{q}{\varepsilon_0} = \int_0^{\rho>R_2} \bar{\rho}\,d\bar{\rho}\int_0^{2\pi} d\phi \int dz\,\frac{q}{\varepsilon_0} = \frac{(k-k)}{\varepsilon_0}\times\int dz \Rightarrow E(\rho>R_2)=0.$$

In the interesting region where the charge inside the surface of integration is nonzero we have

$$R_1 < \rho < R_2 : \iint_{R_1<\rho<R_2} \vec{E}\cdot d\vec{\sigma} = E(\rho)2\pi\rho\Big|_{R_1<\rho<R_2} \int dz$$

$$= \iiint_{R_1<\rho<R_2} d\text{vol}\,\frac{q}{\varepsilon_0} = \int_0^{R_1<\rho<R_2} \bar{\rho}\,d\bar{\rho}\int_0^{2\pi} d\phi \int dz\,\frac{q}{\varepsilon_0} = \frac{k}{\varepsilon_0}\times\int dz$$

$$\Rightarrow \vec{E}(R_1<\rho<R_2) = \frac{k}{2\pi\varepsilon_0}\frac{\hat{\rho}}{\rho}.$$

Finally to write the electric as the gradient of a potential, $\vec{E} = -\vec{\nabla}\varphi$, we can simply integrate. For the inner and outer regions where the electric field vanishes the potential is a constant, and, for simplicity we take the potential to vanish at the center. In the interesting region we need only observe that $-\int d\rho/\rho \Rightarrow \ln(1/\rho) - \text{const}$. Thus with the given boundary condition at $\rho = R_1$, $\varphi(R_1)=0$, we find

$$\rho < R_1 : \vec{E} = 0, \varphi = 0$$

$$R_1 < \rho < R_2 : \vec{E} = \frac{k}{2\pi\varepsilon_0}\frac{\hat{\rho}}{\rho}, \varphi = \frac{k}{2\pi\varepsilon_0}\ln\left(\frac{R_1}{\rho}\right)$$

$$\rho > R_2 : \vec{E} = 0, \varphi = \frac{k}{2\pi\varepsilon_0}\ln\left(\frac{R_1}{R_2}\right).$$

To describe the electric field and the potential right at the surface of the conductor, we should be careful about the (true) thickness of the conductor. Here we take the idealized limit that the conducting cylindrical tube is of zero thickness. Thus the potential is smooth at the conductors, but its gradient, the electric field, changes discontinuously in this limit. We should really be careful about how we define the electric field at the conductors; we must specify whether we are approaching the conductor from inside or outside.

§6.11: 4

Solution: Here we want to practice using curls and Stokes' theorem. We have the vector field $\vec{V} = y\hat{x} + 2\hat{y}$ and a surface composed of the indicated 3 triangles, *i.e.*, the boundary is the remaining triangle in the *x*-*y* plane. To use Stokes' theorem we need to specify the vertices of this triangle. For $z = 0$ the plane defines $2x + 3y = 12$ and the vertices are (6,0,0) and (0,4,0). Thus we have

$$\iint_\sigma \vec{\nabla} \times \left(y\hat{x} + 2\hat{y} \right) \cdot d\vec{\sigma} = \oint_{\text{triangle}} \left(y\hat{x} + 2\hat{y} \right) \cdot d\vec{r}$$

$$= \int_0^6 y\Big|_{y=0} dx + \int_4^0 2dy + \int_6^0 \left(y\hat{x} + 2\hat{y} \right)_{y=2(6-x)/3} \cdot \left(\frac{3\hat{x} - 2\hat{y}}{3} \right) dx$$

$$= 0 - 8 + \int_6^0 \left(\frac{2(6-x)}{3} - \frac{4}{3} \right) dx = -8 + \int_6^0 \left(\frac{8}{3} - \frac{2x}{3} \right) dx$$

$$= -8 + \left(\frac{8x}{3} - \frac{x^2}{3} \right) \Bigg|_6^0 = -8 - 16 + 12 = -12.$$

§6.11: 7

Solution: Now consider the vector field $\vec{V} = \left(x - x^2 z \right) \hat{x} + \left(yz^3 - y^2 \right) \hat{y} + \left(x^2 y - xz \right) \hat{z}$ with the integral of its curl being over any surface with its boundary in the (x,y) plane ($z = 0$). Thus, using Stokes' theorem, we find

$$I = \iint_{\sigma} \left(\vec{\nabla}\times\vec{V}\right)\cdot d\vec{\sigma} = \oint_{\partial\sigma(z=0)} \vec{V}\cdot d\vec{r} = \iint_{\sigma'(z=0)} \left(\vec{\nabla}\times\vec{V}\right)\cdot d\vec{\sigma}.$$

Thus using Stokes' twice we can write the desired integral as a surface integral in the (x,y) plane where $d\vec{\sigma} \propto \hat{z}$. This last result suggests that we evaluate the curl of the vector field in the (x,y) plane. We have

$$\begin{aligned}
\vec{\nabla}\times\vec{V}\Big|_{z=0} &= \left|\begin{matrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x - x^2 z & yz^3 - y^2 & x^2 y - xz \end{matrix}\right|_{z=0} \\
&= \hat{x}\left(x^2 - 3yz^2\right) + \hat{y}\left(-x^2 - 2xy + z\right) + \hat{z}\left(0-0\right)\Big|_{z=0} \\
&= x^2\hat{x} - \left(x^2 + 2xy\right)\hat{y}.
\end{aligned}$$

So we find that $\left(\vec{\nabla}\times\vec{V}\right)\cdot d\vec{\sigma}\Big|_{z=0} = 0$ (the curl as no z component) and thus for any surface with a boundary in the (x,y) plane we have

$$I = \iint_{\sigma} \left(\vec{\nabla}\times\vec{V}\right)\cdot d\vec{\sigma} = \iint_{\sigma'(z=0)} \left(\vec{\nabla}\times\vec{V}\right)\cdot d\vec{\sigma} = 0.$$

§6.11: 10

<u>Solution</u>: Here we again want to practice using Stokes' theorem. We have a vector field $\vec{V} = 2xy\hat{x} + \left(x^2 - 2x\right)\hat{y} - x^2 z^2 \hat{z}$ and a surface defined by $z = 9 - x^2 - 9y^2$. Note that in this exercise we do not have cylindrical symmetry. Instead the intersection of the surface with the x,y plane (z = 0) is an ellipse, $\left(x/3\right)^2 + y^2 = 1$. So first we test for simplifications by evaluating the curl,

$$\vec{\nabla}\times\vec{V}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 2xy & x^2-2x & -x^2z^2\end{vmatrix}=(0)\hat{x}+\left(+2xz^2\right)\hat{y}+(2x-2-2x)\hat{z}=2xz^2\hat{y}-2\hat{z}.$$

We also see that the local normal to the specified surface is

$$\hat{n}=\frac{\vec{\nabla}\left(z+x^2+9y^2\right)}{\left|\vec{\nabla}\left(z+x^2+9y^2\right)\right|}=\frac{2x\hat{x}+18y\hat{y}+\hat{z}}{\sqrt{4x^2+324y^2+1}}.$$

Thus doing the surface integral directly is going to be a bit messy. So instead we use Stokes' theorem on the boundary, which is the ellipse above. On the ellipse we can define the parameterization $x=3\cos\phi$, $y=\sin\phi$ and write

$$\begin{aligned}\iint\limits_{z=9-x^2-9y^2}\left(\vec{\nabla}\times\vec{V}\right)\cdot d\vec{\sigma}&=\oint\limits_{x^2+9y^2=9}\vec{V}\cdot d\vec{s}=\int_0^{2\pi}d\phi\left[\vec{V}\cdot\left(-3\sin\phi\hat{x}+\cos\phi\hat{y}\right)\right]\\ &=\int_0^{2\pi}d\phi\left[-3\sin\phi(2)(3\cos\phi)(\sin\phi)+\cos\phi\left(9\cos^2\phi-6\cos\phi\right)\right]\\ &=3\int_0^{2\pi}d\phi\left[-6\sin^2\phi\cos\phi+\cos^2\phi\left(3\cos\phi-2\right)\right]\\ &=3\int_0^{2\pi}d\phi\left[-6\cos\phi+9\cos^3\phi-2\cos^2\phi\right]\\ &=3\int_0^{2\pi}d\phi\left[-1-\cos 2\phi\right]=-6\pi.\end{aligned}$$

In obtaining the last line we used the fact that the integral of any odd power of the cosine over a full cycle yields zero, plus the double angle formula for the even power, *i.e.*, the average value of $\cos^2\phi$ is $1/2$. We will have more to say about this when we study Fourier series.

To complete this discussion let us try to perform the surface integral directly. We have

$$\left(\vec{\nabla}\times\vec{V}\right)\cdot\hat{n}\sec\gamma=\left(2xz^2\hat{y}-2\hat{z}\right)\cdot\left(\frac{2x\hat{x}+18y\hat{y}+\hat{z}}{\sqrt{4x^2+324y^2+1}}\right)\left(\sqrt{4x^2+324y^2+1}\right)$$
$$=36xyz^2-2=36xy\left(9-x^2-9y^2\right)^2-2$$
$$=36\left(81xy-18x^3y-162xy^3+18x^3y^3+x^5y+81xy^5\right)-2.$$

This looks pretty ugly (with many places to make an arithmetic error). On the other hand, the actual integral expressed in the *x*,*y* plane,

$$\iint_{9=x^2+9y^2} dxdy=\int_{-3}^{3}dx\int_{-\sqrt{1-(x/3)^2}}^{-\sqrt{1-(x/3)^2}}dy,$$

is clearly symmetric in both coordinates. Hence all of the terms with odd powers of *x* and/or *y* (everything but the -2) integrate to zero. So finally we have (recall exercise 6.9:7 above)

$$\iint_{z=9-x^2-9y^2}\left(\vec{\nabla}\times\vec{V}\right)\cdot d\vec{\sigma}=-2\int_{-3}^{3}dx\int_{-\sqrt{1-(x/3)^2}}^{-\sqrt{1-(x/3)^2}}dy=-2\left(\text{area of ellipse}\right)$$
$$=-4\int_{-3}^{3}dx\sqrt{1-(x/3)^2}\xrightarrow[x=3\cos\theta]{}4\int_{\pi}^{0}3\sin^2\theta d\theta$$
$$=4\int_{\pi}^{0}3\frac{1-\cos 2\theta}{2}d\theta=6\theta\Big|_{\pi}^{0}=-6\pi.$$

So we obtained the same answer as above, but we had to work much harder and know more tricks. Clearly using Stokes' theorem is appropriate for smart but lazy physicists.

§6.11: 15

<u>Solution</u>: As a final example consider the contour integral

$$I = \oint_C \left(ydx + zdy + xdz \right),$$

with the contour C defined by the intersection of the surfaces $x + y = 2$ and $x^2 + y^2 + z^2 = 2\left(x + y\right)$. A little thought yields the result that the second surface is a sphere of radius $\sqrt{2}$ centered on the point (1,1,0), $\left(x-1\right)^2 + \left(y-1\right)^2 + z^2 = 2$. Thus the intersection with the first surface (a flat surface parallel to the *z*-axis) is a circle of radius $\sqrt{2}$ centered on the point (1,1,0). To simplify things we evaluate the curl of the corresponding vector field, $\vec{V} = y\hat{x} + z\hat{y} + x\hat{z}$ to find

$$\vec{\nabla} \times \vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = \left(-1\right)\hat{x} + \left(-1\right)\hat{y} + \left(-1\right)\hat{z} = -\left(\hat{x} + \hat{y} + \hat{z}\right).$$

Thus the curl is a simple, constant vector and we should be able to Stokes' theorem. We find

$$I = \oint_C \left(ydx + zdy + xdz \right) = \oint_{\partial\sigma} \vec{V} \cdot d\vec{r} = \iint_{\sigma} \left(\vec{\nabla} \times \vec{V} \right) \cdot d\vec{\sigma}$$
$$= -\left(\hat{x} + \hat{y} + \hat{z}\right) \cdot \hat{n}\, A_\sigma = -\left(\hat{x} + \hat{y} + \hat{z}\right) \cdot \hat{n}\left(2\pi\right).$$

So the final question is to find the normal to the plane of the contour. This is just the normal to the plane $x + y = 2$, $\hat{n} = \left(\hat{x} + \hat{y}\right)\big/\sqrt{2}$, where we have used our previous knowledge that the equation of a plane is given by $\left(\vec{r} - \vec{r}_0\right) \cdot \hat{n} = 0$ (with $\vec{r}_0 = \hat{x} + \hat{y}$). So finally we have (note that the sign here is actually ambiguous since the text does not specify the sense of the original contour)

$$I = -\left(\hat{x} + \hat{y} + \hat{z}\right) \cdot \frac{\left(\hat{x} + \hat{y}\right)}{\sqrt{2}} \left(2\pi\right) = -2\sqrt{2}\pi.$$