Lecture 14 – Appendix B: Some sample problems from Boas

Here are some solutions to the sample problems assigned for Chapter 8.

§8.2: 6

<u>Solution</u>: We want to find the solution to the following first order equation using separation of variables. We find

$$y' = \frac{2xy^2 + x}{x^2 y - y} \Rightarrow y'y\left(x^2 - 1\right) = x\left(2y^2 + 1\right) \Rightarrow \int \frac{dy\, y}{\left(2y^2 + 1\right)} = \int \frac{dx\, x}{\left(x^2 - 1\right)}$$

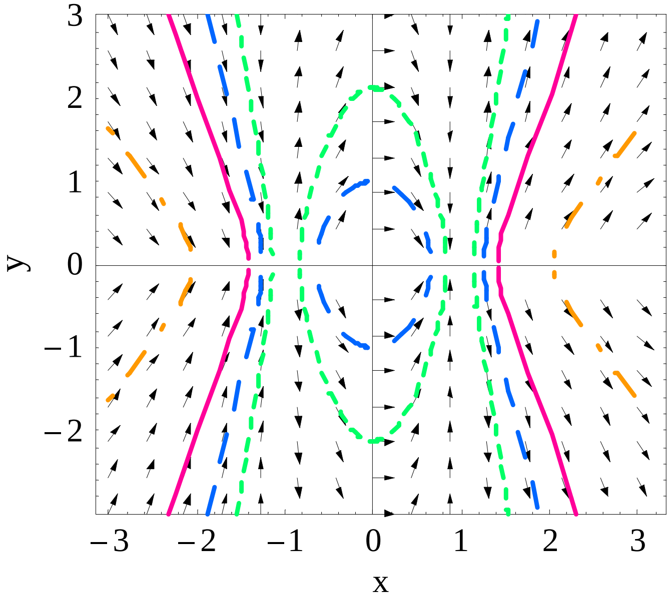
$$\Rightarrow \frac{1}{4}\ln\left(2y^2 + 1\right) = \frac{1}{2}\ln\left(x^2 - 1\right) + c \Rightarrow 2y^2 + 1 = C\left(x^2 - 1\right)^2$$

$$\Rightarrow y(x) = \pm\sqrt{\frac{C\left(x^2 - 1\right)^2 - 1}{2}}.$$

We can check by taking a derivative and substituting back into the original equation. To match the boundary condition, $y\left(\sqrt{2}\right) = 0$, we find

$$y\left(\sqrt{2}\right) = \pm\sqrt{\frac{C\left(2-1\right)^2 - 1}{2}} = \pm\sqrt{\frac{C-1}{2}} = 0 \Rightarrow C = 1.$$

Using *Mathematica* we find the figure to the right indicating the slope field and curves for $C = 1$ (solid, red curve), $C = 3$ (long dashed, blue curve), $C = 10$ (short dashed, green curve) and $C = 0.1$ (dot-dashed, orange curve). In each case both branches of the square root are plotted. The solid curve goes through $\left(\sqrt{2}, 0\right)$ as desired.



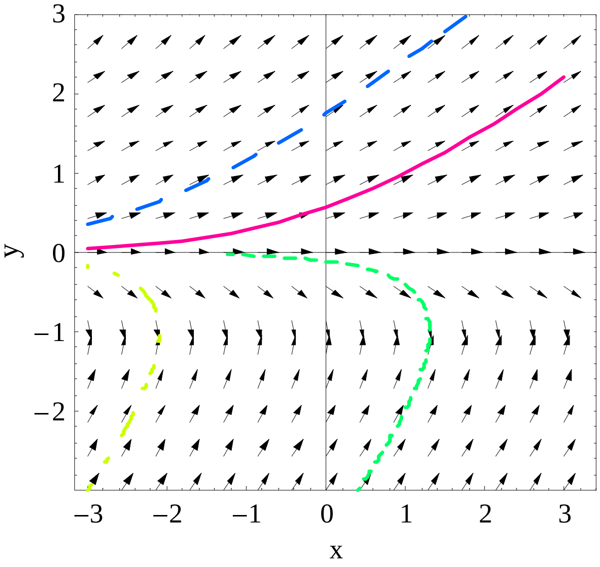
§8.2: 9

Solution: Now for a slightly different equation we find

$$(1+y)y' = y \Rightarrow \int \frac{dy(1+y)}{y} = \int dy\left(1+\frac{1}{y}\right) = \int dx$$
$$\Rightarrow y + \ln y = x + c \Rightarrow ye^{y} = Ce^{x}.$$

Here check use implicit differentiation. For the particular solution satisfying $y(1)=1$ we require

$$1e^{1} = Ce^{1} \Rightarrow C = 1.$$



Using *Mathematica* we find the figure to the right indicating the slope field and curves for $C=1$ (solid, red curve), $C=10$ (long dashed, blue curve), $C=-0.1$ (short dashed, green curve) and $C=-3$ (dot-dashed, yellow-green curve). In each case both branches of the square root are plotted. The solid curve goes through $(1,1)$ as desired. Note that $y(x)$ is double valued for $C < 0$ and care must be taking to plot the function, e.g., evaluate $x(y)$ instead.

§8.2: 13

Solution: We want to study the equation given in exercise 8.2:2. First we find the standard solution found by separation of variables. We have

$$x\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0 \Rightarrow \int \frac{xdx}{\sqrt{1-x^2}} + \int \frac{ydy}{\sqrt{1-y^2}} = 0$$
$$\Rightarrow \sqrt{1-x^2} + \sqrt{1-y^2} = -C \Rightarrow y = \pm\sqrt{1-\left(C+\sqrt{1-x^2}\right)^2}.$$

Since we have divided by the square roots, this general form for the solution misses the singular solutions given by $y = \pm 1, x = \pm 1$, where the square roots both vanish.

§8.3: 3

Solution: We start with this first order linear differential equation in standard form

$$dy + \left(2xy - xe^{-x^2}\right)dx \Rightarrow y' + (2x)y = xe^{-x^2}$$
$$\Rightarrow P = 2x, Q = xe^{-x^2} \Rightarrow I = \int Pdx = x^2.$$
$$\Rightarrow y(x)e^{x^2} = \int dx\, x e^{-x^2} e^{x^2} + c = \frac{x^2}{2} + c$$
$$\Rightarrow y(x) = \left(\frac{x^2}{2} + c\right)e^{-x^2}.$$

If we use *Mathematica* to solve the equation, we find

```
DSolve[y'[x]+2x*y[x]==x*Exp[-x^2],y[x],x]
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$$\left\{\left\{y[x] \to \frac{1}{2} e^{-x^2} x^2 + e^{-x^2} C[1]\right\}\right\}$$

So in this case we get the same result as via the analytic approach above.

§8.3: 15

Solution: We want to study the salt flow problem defined by this exercise. We define $S(t)$ as the salt content (in pounds) in the lake at time *t* hours ($S(0) = 10^7$ lb). We also define the volume of the lake at time *t* as $V(t)$ ($V(0) = 10^9$ gal). The description of flow of water in and out yields the following pair of first order differential equations

$$\dot{V}(t) = 4\times10^{5}\left(\frac{\text{gal}}{\text{hr}}\right) - 10^{5}\left(\frac{\text{gal}}{\text{hr}}\right) \Rightarrow V(t) = V(0) + 3\times10^{5}\left(\frac{\text{gal}}{\text{hr}}\right)t$$

$$\Rightarrow V(t) = \left[10^{9} + 3\times10^{5}\left(\frac{t}{\text{hr}}\right)\right]\ \text{gal} = 10^{9}\left[1 + 3\times10^{-4}\left(\frac{t}{\text{hr}}\right)\right]\ \text{gal},$$

$$\dot{S}(t) = 4\times10^{5}\left(\frac{\text{gal}}{\text{hr}}\right)\times\left(\frac{5\ \text{lb}}{1000\ \text{gal}}\right) - \frac{S(t)}{V(t)}\times10^{5}\left(\frac{\text{gal}}{\text{hr}}\right)$$

$$\Rightarrow \dot{S}(t) + S(t)\left[\frac{10^{-4}\left(\frac{1}{\text{hr}}\right)}{1 + 3\times10^{-4}\left(\frac{t}{\text{hr}}\right)}\right] = 2\times10^{3}\left(\frac{\text{lb}}{\text{hr}}\right).$$

The second, nontrivial equation we solve as in the previous exercise. We find (ignoring the units until the end)

$$P = \left[\frac{10^{-4}}{1 + 3\times10^{-4}t}\right],\ Q = 2\times10^{3}$$

$$\Rightarrow I = 10^{-4}\int\frac{dt}{1 + 3\times10^{-4}t} = \frac{\ln\left(1 + 3\times10^{-4}t\right)}{3} \Rightarrow e^{I} = \sqrt[3]{1 + 3\times10^{-4}t}$$

$$\Rightarrow S(t)\sqrt[3]{1 + 3\times10^{-4}t} = \int 2\times10^{3}\,dt\sqrt[3]{1 + 3\times10^{-4}t} + c = 2\times10^{3}\left[\frac{\left(1 + 3\times10^{-4}t\right)^{4/3}}{4/3\left(3\times10^{-4}\right)}\right] + c$$

$$\Rightarrow S(t) = 0.5\times10^{7}\left(1 + 3\times10^{-4}t\right) + \frac{c}{\sqrt[3]{1 + 3\times10^{-4}t}}$$

$$\Rightarrow S(t) = 0.5\times10^{7}\ \text{lb}\left[\left(1 + 3\times10^{-4}\frac{t}{\text{hr}}\right) + \frac{1}{\sqrt[3]{1 + 3\times10^{-4}\frac{t}{\text{hr}}}}\right].$$

The last step made use of the initial condition on the amount of salt in the lake.

§8.3: 17

Solution: Here we want to consider an RC circuit described by the first order equation

$$R\dot{I}+\frac{I}{C}=\frac{dV}{dt}=-\omega V_0\sin\omega t\Rightarrow \dot{I}+\frac{I}{RC}=-\omega V_0\sin\omega t.$$

In terms of the methods used here we have

$$\begin{aligned}&P=\frac{1}{RC},\, Q=-\omega\frac{V_0}{R}\sin\omega t\\ &\Rightarrow I\left(t\right)e^{t/RC}=-\omega\frac{V_0}{R}\int dt\sin\omega t\, e^{t/RC}+A.\end{aligned}$$

The challenging bit here is performing the integral, which we accomplish with a double integration by parts and then recombining and solving,

$$\begin{aligned}\int dt\sin\omega t\, e^{t/RC}&=RC\sin\omega te^{t/RC}-\int dt\,\omega RC\cos\omega te^{t/RC}\\ &=RC\sin\omega te^{t/RC}-\omega R^2C^2\cos\omega te^{t/RC}-\int dt\,\omega^2R^2C^2\sin\omega te^{t/RC}\\ \Rightarrow\left(1+R^2C^2\omega^2\right)\int dt\sin\omega t\, e^{t/RC}&=RC\left[\sin\omega t-\omega RC\cos\omega t\right]e^{t/RC}\\ \Rightarrow\int dt\sin\omega t\, e^{t/RC}&=\frac{RC\left[\sin\omega t-\omega RC\cos\omega t\right]e^{t/RC}}{1+R^2C^2\omega^2}.\end{aligned}$$

So finally we have

$$I\left(t\right)=Ae^{-t/RC}-\frac{\omega V_0C\left[\sin\omega t-\omega RC\cos\omega t\right]}{1+R^2C^2\omega^2},$$

where the first term is the solution to the homogeneous equations and the second is a particular solution.

§8.5: 5

Solution: We consider the homogenous linear 2nd order differential equation

$$\left(D^2-2D+1\right)y=\left(D-1\right)^2 y=0.$$

With our usual Ansatz $y=y_0 e^{\alpha x}$ we quickly find

$$\left(\alpha-1\right)^2 y_0 e^{\alpha x}=0 \Rightarrow \alpha_1=\alpha_2=1\left[y_0\neq 0\right]$$
$$\Rightarrow y\left(x\right)=\left(Ax+B\right)e^x.$$

Using *Mathematica* we find easily

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DSolve[y''[x]-2*y'[x]+y[x]==0,y[x],x]
  {{y[x] → ⅇx C[1] + ⅇx x C[2]}}
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which differs only in the labels for the constants of integration.

§8.5: 20

Solution: Here we consider a homogeneous linear 2nd order differential equation with complex coefficients, where we can proceed as above but with complex results. With the same Ansatz we have

$$y''+\left(1-i\right)y'-iy=0\Rightarrow\left[\alpha^2+\left(1-i\right)\alpha-i\right]y_0 e^{\alpha x}=0$$
$$\Rightarrow\alpha^2+\left(1-i\right)\alpha-i=0$$
$$\Rightarrow\alpha_{1,2}=-\frac{\left(1-i\right)}{2}\pm\sqrt{\frac{\left(1-i\right)^2}{4}+i}=-\frac{\left(1-i\right)}{2}\pm\sqrt{\frac{i}{2}}=-\frac{\left(1-i\right)}{2}\pm\frac{1+i}{2}=-1,i$$
$$y\left(x\right)=Ae^{-x}+Be^{ix}.$$

In doing the complex arithmetic we used that fact that $\sqrt{i}=e^{i\pi/4\pm i\pi n}=\pm\left(1+i\right)\big/\sqrt{2}$.

§8.6: 13

Solution: Now we consider the inhomogeneous linear 2nd order differential equation

$$\left(D^2-2D+1\right)y=2\cos x.$$

We have already obtained the complementary solution to the homogenous equation in Exercise 8.5:5. We need only find a particular solution. To that end we consider the complex equation $\left(D^2-2D+1\right)z=2e^{ix}$ and try the Ansatz $z=z_0e^{ix}$. Thus we find, taking the real part in the end,

$$\left(i-1\right)^2 z_0e^{ix}=2e^{ix}\Rightarrow z_0=\frac{2}{\left(i-1\right)^2}=\frac{2}{-2i}=i$$

$$\Rightarrow y_p\left(x\right)=\mathrm{Re}\left[ie^{ix}\right]=-\sin x.$$

It is easy to verify that this result solves the original equation, $\left(D^2-2D+1\right)\left(-\sin x\right)=2\cos x$. Thus the general solution to the equation is

$$y\left(x\right)=\left(Ax+B\right)e^x-\sin x.$$

Using *Mathematica* we find the same result,

```
DSolve[y''[x]-2*y'[x]+y[x]==2*Cos[x],y[x],x]
  {{y[x] → ⅇ^x C[1] + ⅇ^x x C[2] - Sin[x]}}
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§8.6: 26

Solution: Now consider the equation

$$\left(D^2+1\right)y=8x\sin x.$$

With the usual Ansatz we find the complementary solution from $\left(\alpha^2+1\right)=0$, $\alpha_{1,2}=\pm i$ or $y_c=A\sin x+B\cos x$. Since the complementary solution has the same structure as the inhomogeneous term, to find the particular solution we try a complex Ansatz of the form $z=\left(z_1x^2+z_2x\right)e^{iz}$ and solve (looking for the imaginary part)

$$\left(D^2+1\right)z = z_1\left(i^2x^2+4ix+2+x^2\right)e^{ix} + z_2\left(i^2x+2i+x\right)e^{ix} = 8xe^{ix}$$
$$\Rightarrow z_1\left(4ix+2\right)+z_2\left(2i\right)=8x \Rightarrow z_1=-2i,\, z_2=2$$
$$\Rightarrow y_p = \mathrm{Im}\left[\left(-2ix^2+2x\right)e^{ix}\right] = -2x^2\cos x + 2x\sin x.$$

Thus the general solution is given by

$$y\left(x\right) = A\sin x + B\cos x - 2x\left(x\cos x - \sin x\right).$$

§8.6: 33

Solution: Now we consider a case with several terms on the right-hand-side of the linear 2nd order inhomogeneous differential equation and make more explicit use of linear superposition. First consider the homogenous equation where our previous experience yields

$$y'' + y = 0 \Rightarrow y_c = A\sin x + B\cos x \text{ or } A\sin\left(x+\gamma\right).$$

Now we address each term on the right-hand-side separately using the usual techniques to find the corresponding particular solutions:

$$y'' + y = x^3 - 1 \Rightarrow y_{p1} = ax^3 + bx^2 + cx + d$$
$$\Rightarrow a\left(6x + x^3\right) + b\left(2 + x^2\right) + cx + d = x^3 - 1$$
$$\Rightarrow a = 1, b = 0, c = -6, d = -1$$
$$\Rightarrow y_{p1} = x^3 - 6x - 1;$$
$$y'' + y = 2\cos x \Rightarrow z = z_0xe^{ix} : z'' + z = 2e^{ix}$$
$$\Rightarrow z_0\left(-x + 2i + x\right)e^{ix} = 2e^{ix} \Rightarrow z_0 = -i$$
$$\Rightarrow y_{p2} = \mathrm{Re}\left[-ixe^{ix}\right] = x\sin x;$$

$$
\begin{aligned}
&y'' + y = (2-4x)e^{x} \Rightarrow y_{p3} = (ax+b)e^{x}\\
&\Rightarrow a(2+x+x)e^{x} + b(1+1)e^{x} = (2-4x)e^{x}\\
&\Rightarrow a = -2, b = 3\\
&\Rightarrow y_{p3} = (3-2x)e^{x}.
\end{aligned}
$$

Pulling it all together we have the general solution in the form

$$y(x) = A\sin x + B\cos x + x^{3} - 6x - 1 + x\sin x + (3-2x)e^{x}.$$

As usual this result can be checked by substituting into the original equation.

§8.6: 34

Solution: Again we use linear superposition, here on the equation $y'' - 5y' + 6y = 2e^{x} + 6x - 5$. First consider the homogeneous equation and the complementary solution. We have

$$
\begin{aligned}
&y'' - 5y' + 6y = 0 \Rightarrow y_{c} \propto e^{\alpha x} \Rightarrow \alpha_{\pm} = \frac{5 \pm \sqrt{25-24}}{2} = 2,3\\
&\Rightarrow y_{c} = Ae^{2x} + Be^{3x}.
\end{aligned}
$$

Now we find the particular solutions from

$$
\begin{aligned}
&y'' - 5y' + 6y = 2e^{x} \Rightarrow y_{p1} = ae^{x}\\
&\Rightarrow ae^{x}(1-5+6) = 2ae^{x} = 2e^{x} \Rightarrow a = 1\\
&\Rightarrow y_{p1} = e^{x},\\
&y'' - 5y' + 6y = 6x - 5 \Rightarrow y_{p2} = ax + b\\
&\Rightarrow (-5+6x)a + 6b = 6x - 5 \Rightarrow a = 1, b = 0\\
&\Rightarrow y_{p2} = x.
\end{aligned}
$$

Thus, via linear superposition, the full solution is,

$$y = Ae^{2x} + Be^{3x} + e^{x} + x.$$

§8.6: 37

Solution: Finally one more exam of an inhomogeneous linear 2nd order differential equation, but now with some degeneracy. First the complementary solution is (note the form of the second solution in this degenerate case)

$$\begin{aligned}&(D-1)^2 y = 0 \Rightarrow y_c \propto e^{\alpha x} \Rightarrow (\alpha - 1)^2 = 0 \Rightarrow \alpha_{\pm} = 1\\ &\Rightarrow y_c = Ae^{x} + Bxe^{x}.\end{aligned}$$

Now we find the particular solutions, where we have one driving function that is identical to a complementary solution (see Eq. 6.24 in Boas). We have

$$\begin{aligned}&(D-1)^2 y = \left(D^2 - 2D + 1\right) y = 4e^{x} \Rightarrow y_{p1} = ax^2e^{x}\\ &\Rightarrow ae^{x}\left(2 + 4x + x^2 - 4x - 2x^2 + x^2\right) = 2ae^{x} = 4e^{x} \Rightarrow a = 2\\ &\Rightarrow y_{p1} = 2x^2e^{x},\\ &(D-1)^2 y = (1-x)e^{2x} \Rightarrow y_{p2} = (ax+b)e^{2x}\\ &\Rightarrow e^{2x}\left[(4x + 4 - 4x - 2 + x)a + b\right] = e^{2x}\left[(x+2)a + b\right] = (1-x)e^{2x}\\ &\Rightarrow a = -1, b = 3\\ &\Rightarrow y_{p2} = (3-x)e^{2x},\\ &(D-1)^2 y = x - 1 \Rightarrow y_{p3} = ax + b\\ &\Rightarrow -2a + ax + b = x - 1 \Rightarrow a = 1, b = 1\\ &\Rightarrow y_{p3} = x + 1.\end{aligned}$$

Thus the full solution is the sum

$$y = Ae^{x} + Bxe^{x} + 2x^{2}e^{x} + \left(3-x\right)e^{2x} + x + 1.$$

§8.13: 3

Solution: We start with the equation $y''' + 2y'' + 2y' = 0$, which we identify as a 2nd order linear homogeneous differential equation for y'. We can solve for y' in the usual way with an exponential Ansatz, $y' = Ae^{\alpha x}$,

$$\begin{aligned}
&y''' + 2y'' + 2y' = \left(\alpha^{2} + 2\alpha + 2\right)Ae^{\alpha x} = 0 \\
&\Rightarrow \alpha = -1 \pm \sqrt{1-2} = -1 \pm i \\
&\Rightarrow y'\left(x\right) = e^{-x}\left(A\sin x + B\cos x\right).
\end{aligned}$$

To find the original function we integrate one more which allows also a constant solution (note the 3 constants of integration),

$$y\left(x\right) = C + e^{-x}D\cos\left(x+\phi\right).$$

§8.13: 13

Solution: We have the equation $y'' + 4y' + 5y = 26e^{3x}$, which we identify as a 2nd order linear inhomogeneous differential equation for y. As in the earlier exercises we start with the complementary solution and then proceed to find appropriate particular solutions with a form suggested by the inhomogeneous term. We have

$$\begin{aligned}
&y'' + 4y' + 5y = 0 \Rightarrow y_{c} \propto e^{\alpha x} \Rightarrow \alpha^{2} + 4\alpha + 5 = 0 \\
&\Rightarrow \alpha_{\pm} = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm 2i \\
&\Rightarrow y_{c} = Ae^{-2x}\cos\left(2x+\phi\right),
\end{aligned}$$

and

$$y'' + 4y' + 5y = 26e^{3x} \Rightarrow y_p = ae^{3x}$$
$$\Rightarrow ae^{3x}\left(9 + 12 + 5\right) = 26ae^{3x} = 26e^{3x} \Rightarrow a = 1$$
$$\Rightarrow y_p = e^{3x}$$
$$\Rightarrow y = Ae^{-2x}\cos\left(2x + \phi\right) + e^{3x} = Be^{-2x}\sin\left(2x + \gamma\right) + e^{3x}.$$

§8.13: 26

Solution: Here we want to solve an equation and find a particular solution satisfying a boundary condition. The equation is 1st order and separable with an integrating factor. We find by the standard techniques

$$xy' - y = x^2 \Rightarrow y' - \frac{y}{x} = x \Rightarrow P = -\frac{1}{x}, Q = x$$
$$\Rightarrow I = -\int \frac{dx}{x} = -\ln x \Rightarrow e^P = \frac{1}{x}$$
$$\Rightarrow \frac{y\left(x\right)}{x} = \int dx\, x \frac{1}{x} + c = x + c$$
$$\Rightarrow y\left(x\right) = x^2 + cx : \left[y\left(2\right) = 6 \Rightarrow c = 1\right]$$
$$\Rightarrow y_p\left(x\right) = x^2 + x.$$