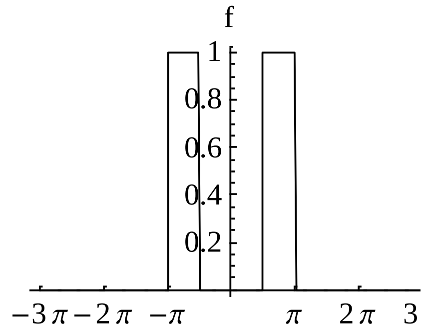
Lecture 16 – Appendix B: Some sample problems from Boas

Here are some solutions to the sample problems assigned for Chapter 7.12.

§7.12: 4



Solution: We want to find the Fourier transform of the non-periodic, even function defined by

$$f(x) = \begin{cases} 1, & \pi/2 < |x| < \pi \\ 0, & \text{otherwise} \end{cases}.$$

Proceeding to find the transform we have

$$\begin{aligned}
g(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x) e^{-i\alpha x} = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{-\pi/2} dx\, e^{-i\alpha x} + \frac{1}{\sqrt{2\pi}} \int_{\pi/2}^{\pi} dx\, e^{-i\alpha x} \\
&= \frac{1}{\sqrt{2\pi}(-i\alpha)} \left\{ e^{-i\alpha x}\Big|_{-\pi}^{-\pi/2} + e^{-i\alpha x}\Big|_{\pi/2}^{\pi} \right\} \\
&= \frac{1}{\sqrt{2\pi}(-i\alpha)} \left\{ e^{i\alpha\pi/2} - e^{i\alpha\pi} + e^{-i\alpha\pi} - e^{-i\alpha\pi/2} \right\} \\
&= \sqrt{\frac{2}{\pi}} \frac{\sin(\alpha\pi) - \sin(\alpha\pi/2)}{\alpha}.
\end{aligned}$$

Thus, transforming back to the function of x, we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\alpha\, g(\alpha) e^{i\alpha x} = \int_{-\infty}^{\infty} d\alpha \frac{\sin(\alpha\pi) - \sin(\alpha\pi/2)}{\pi\alpha} e^{i\alpha x}.$$

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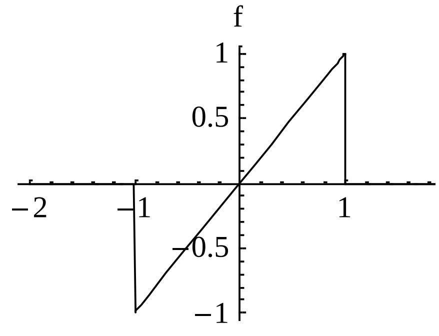


Solution: Now consider the odd function in the figure defined by

$$f(x) = \begin{cases} x, & |x| < 1 \\ 0, & |x| > 1 \end{cases}.$$

Proceeding to find the transform we have

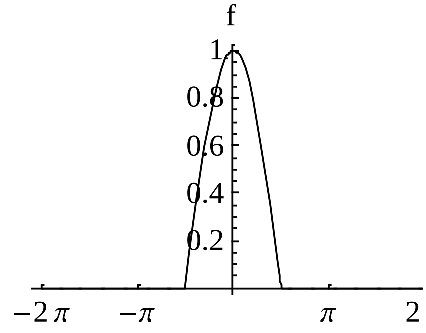
$$\begin{aligned} g(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x) e^{-i\alpha x} = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} dx\, x\, e^{-i\alpha x} \\ &= \frac{1}{\sqrt{2\pi}(-i\alpha)} \left\{ x e^{-i\alpha x} \Big|_{-1}^{1} \right\} + \frac{1}{\sqrt{2\pi}(i\alpha)} \int_{-1}^{1} dx\, e^{-i\alpha x} \\ &= \frac{1}{\sqrt{2\pi}(-i\alpha)} \left\{ e^{-i\alpha} + e^{i\alpha} \right\} + \frac{1}{\sqrt{2\pi}(\alpha^2)} \left\{ e^{-i\alpha x} \Big|_{-1}^{1} \right\} \\ &= \sqrt{\frac{2}{\pi}} \frac{\cos(\alpha) i}{\alpha} - \sqrt{\frac{2}{\pi}} \frac{\sin(\alpha) i}{\alpha^2}. \end{aligned}$$

Thus, transforming back to the function of *x*, we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\alpha\, g(\alpha) e^{i\alpha x} = \frac{i}{\pi} \int_{-\infty}^{\infty} d\alpha \frac{\alpha \cos(\alpha) - \sin(\alpha)}{\alpha^2} e^{i\alpha x}.$$

§7.12: 11

Solution: Now consider the even function



$$f(x) = \begin{cases} \cos x, & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0, & |x| > \frac{\pi}{2} \end{cases}.$$

Proceeding to find the transform we have

$$g(\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dx\, f(x)e^{-i\alpha x}=\frac{1}{\sqrt{2\pi}}\int_{-\pi/2}^{\pi/2}dx\cos x\, e^{-i\alpha x}$$

$$=\frac{1}{2\sqrt{2\pi}}\int_{-\pi/2}^{\pi/2}dx\; e^{-i\alpha x+ix}+e^{-i\alpha x-ix}$$

$$=\frac{1}{2\sqrt{2\pi}}\left\{\left.\frac{e^{-i\alpha x+ix}}{-i(\alpha-1)}\right|_{-\pi/2}^{\pi/2}+\left.\frac{e^{-i\alpha x-ix}}{-i(\alpha+1)}\right|_{-\pi/2}^{\pi/2}\right\}$$

$$=\frac{i}{2\sqrt{2\pi}}\left\{\frac{e^{-i(\alpha-1)\pi/2}-e^{i(\alpha-1)\pi/2}}{\alpha-1}+\frac{e^{-i(\alpha+1)\pi/2}-e^{i(\alpha+1)\pi/2}}{\alpha+1}\right\}$$

$$=\frac{i}{2\sqrt{2\pi}}\left\{\frac{ie^{-i\alpha\pi/2}+ie^{i\alpha\pi/2}}{\alpha-1}+\frac{-ie^{-i\alpha\pi/2}-ie^{i\alpha\pi/2}}{\alpha+1}\right\}$$

$$=\frac{1}{\sqrt{2\pi}}\cos\left(\alpha\frac{\pi}{2}\right)\left(\frac{-1}{\alpha-1}+\frac{1}{\alpha+1}\right)=-\sqrt{\frac{2}{\pi}}\frac{\cos(\alpha\pi/2)}{\alpha^2-1}.$$

Thus, transforming back to the function of *x*, we have

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\alpha\, g(\alpha)e^{i\alpha x}=\frac{1}{\pi}\int_{-\infty}^{\infty}d\alpha\frac{\cos(\alpha\pi/2)}{1-\alpha^2}e^{i\alpha x}.$$

§7.12: 16

<u>Solution</u>: Here we consider the even function of the previous exercise and use the Fourier cosine transform. We find

$$g_c(\alpha)=\sqrt{\frac{2}{\pi}}\int_0^\infty dx\, f(x)\cos(\alpha x)=\sqrt{\frac{2}{\pi}}\int_0^{\pi/2} dx\cos x\cos(\alpha x)$$

$$=\frac{1}{2\sqrt{2\pi}}\int_0^{\pi/2} dx\left(e^{i\alpha x+ix}+e^{i\alpha x-ix}+e^{-i\alpha x+ix}+e^{-i\alpha x-ix}\right)$$

$$=\frac{1}{2\sqrt{2\pi}}\left\{\left.\frac{e^{i\alpha x+ix}}{i(\alpha+1)}\right|_0^{\pi/2}+\left.\frac{e^{i\alpha x-ix}}{i(\alpha-1)}\right|_0^{\pi/2}+\left.\frac{e^{-i\alpha x+ix}}{-i(\alpha-1)}\right|_0^{\pi/2}+\left.\frac{e^{-i\alpha x-ix}}{-i(\alpha+1)}\right|_0^{\pi/2}\right\}$$

$$=\frac{i}{2\sqrt{2\pi}}\left\{\frac{e^{-i(\alpha-1)\pi/2}-e^{i(\alpha-1)\pi/2}}{\alpha-1}+\frac{e^{-i(\alpha+1)\pi/2}-e^{i(\alpha+1)\pi/2}}{\alpha+1}\right\}$$

$$=\frac{i}{2\sqrt{2\pi}}\left\{\frac{ie^{-i\alpha\pi/2}+ie^{i\alpha\pi/2}}{\alpha-1}+\frac{-ie^{-i\alpha\pi/2}-ie^{i\alpha\pi/2}}{\alpha+1}\right\}$$

$$=\frac{1}{\sqrt{2\pi}}\cos\left(\alpha\frac{\pi}{2}\right)\left(\frac{-1}{\alpha-1}+\frac{1}{\alpha+1}\right)=-\sqrt{\frac{2}{\pi}}\frac{\cos(\alpha\pi/2)}{\alpha^2-1}.$$

Thus, transforming back to the function of *x*, we have

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty d\alpha\, g_c(\alpha)\cos(\alpha x)=\frac{2}{\pi}\int_0^\infty d\alpha\frac{\cos(\alpha\pi/2)}{1-\alpha^2}\cos(\alpha x).$$

§7.12: 18

Solution: Next consider the Fourier sine transform applied to the odd function of exercise 7.12:6. We have

$$g_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty dx\, f(x) \sin(\alpha x) = \sqrt{\frac{2}{\pi}} \int_0^1 dx\, x \sin(\alpha x)$$

$$= \sqrt{\frac{2}{\pi}} \left\{ x \frac{-\cos(\alpha x)}{\alpha} \Bigg|_0^1 \right\} + \sqrt{\frac{2}{\pi}} \int_0^1 dx \frac{\cos(\alpha x)}{\alpha}$$

$$= -\sqrt{\frac{2}{\pi}} \frac{\cos(\alpha)}{\alpha} + \sqrt{\frac{2}{\pi}} \frac{1}{(\alpha^2)} \left\{ \sin(\alpha x)\Big|_0^1 \right\}$$

$$= -\sqrt{\frac{2}{\pi}} \frac{\cos(\alpha)}{\alpha} + \sqrt{\frac{2}{\pi}} \frac{\sin(\alpha)}{\alpha^2} = \sqrt{\frac{2}{\pi}} \frac{\sin(\alpha) - \alpha\cos(\alpha)}{\alpha^2}.$$

Thus, transforming back to the function of *x*, we have

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty d\alpha\, g_s(\alpha) \sin(\alpha x)$$

$$= \frac{2}{\pi} \int_0^\infty d\alpha \frac{\sin(\alpha) - \alpha\cos(\alpha)}{\alpha^2} \sin(\alpha x).$$

§7.12: 22

Solution: Using the result of the previous exercise we can evaluate the following integral of the spherical Bessel function of the first kind. We have

$$\int_0^\infty d\alpha\, j_1(\alpha) \sin(\alpha x) = \int_0^\infty d\alpha \frac{\sin(\alpha) - \alpha\cos(\alpha)}{\alpha^2} \sin(\alpha x)$$

$$= \frac{\pi}{2} f_{7.12:6}(x) = \begin{cases} \dfrac{\pi x}{2}, & |x| < 1 \\ 0, & |x| > 1 \end{cases}.$$