Lecture 19 – Appendix B: Some sample problems from Boas

Here are some solutions to the sample problems using delta functions and Green functions (Chapter 8).

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<u>Solution</u>: Now we want to consider the response to a delta function driving function. We consider the equation (and assume homogeneous initial conditions)

$$\ddot{y}+2\dot{y}+y=\delta\left(t-t_0\right),\left[y\left(0\right)=\dot{y}\left(0\right)=0\right].$$

We proceed by taking transforms to find

$$\begin{aligned}
&\mathfrak{L}\left[\dot{y}\right]=p\mathfrak{L}\left[y\right]-y\left(0\right)=p\mathfrak{L}\left[y\right],\\
&\mathfrak{L}\left[\ddot{y}\right]=p\mathfrak{L}\left[\dot{y}\right]-\dot{y}\left(0\right)=p^2\mathfrak{L}\left[y\right]-py\left(0\right)-\dot{y}\left(0\right)=p^2\mathfrak{L}\left[y\right],\\
&\Rightarrow\mathfrak{L}\left[\ddot{y}+2\dot{y}+y=\delta\left(t-t_0\right)\right]\\
&\qquad=p^2\mathfrak{L}\left[y\right]+2p\mathfrak{L}\left[z\right]+\mathfrak{L}\left[y\right]=\mathfrak{L}\left[\delta\left(t-t_0\right)\right]=e^{-pt_0}\\
&\Rightarrow\mathfrak{L}\left[y\right]=\frac{e^{-pt_0}}{p^2+2p+1}=\frac{e^{-pt_0}}{\left(p+1\right)^2}=e^{-pt_0}\mathfrak{L}\left[te^{-t}\right].
\end{aligned}$$

This expression we can easily invert to find the response to the unit impulse using care with the extra exponential (see L28 on page 470 in Boas)

$$y\left(t\right)=\begin{cases}\left(t-t_0\right)e^{-\left(t-t_0\right)}:t>t_0\\ 0:t<t_0\end{cases}.$$

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<u>Solution</u>: Now consider using the delta function to express localized (*i.e.*, point) distribution. a) a mass of 5 at x = 2 and a mass 3 at x = -7 can be conveniently

expressed as $5\delta(x-2)+3\delta(x+7)$. b) a charge 3 at x = -5 and a charge -4 at x = 10 can be conveniently expressed as $3\delta(x+5)-4\delta(x-10)$.

§8.11: 15

Solution: We want to evaluate integrals involving delta functions being careful about its defining properties (and the presence of implied theta functions/step functions which we make explicit here). We have

$$\text{(a)} \int_0^\pi dx \sin x \delta\left(x-\frac{\pi}{2}\right) = \sin\frac{\pi}{2}\Theta\left(\pi-\frac{\pi}{2}\right)\Theta\left(\frac{\pi}{2}-0\right) = 1,$$

$$\text{(b)} \int_0^\pi dx \sin x \delta\left(x+\frac{\pi}{2}\right) = \sin\left(-\frac{\pi}{2}\right)\Theta\left(\pi+\frac{\pi}{2}\right)\Theta\left(-\frac{\pi}{2}-0\right) = 0,$$

$$\begin{aligned}\text{(c)} \int_{-1}^1 dx e^{3x}\delta'(x) &= e^{3x}\delta(x)\Big|_{-1}^1 - 3\int_{-1}^1 dx e^{3x}\delta(x) \\ &= -3e^0\Theta(1-0)\Theta(0-(-1)) = -3,\end{aligned}$$

$$\begin{aligned}\text{(d)} \int_0^\pi dx \cosh x \delta''(x-1) &= \cosh x \delta'(x-1)\Big|_0^\pi - \int_0^\pi dx \sinh x \delta'(x-1) \\ &= -\sinh x \delta(x-1)\Big|_0^\pi + \int_0^\pi dx \cosh x \delta(x-1) \\ &= (\cosh 1)\Theta(\pi-1)\Theta(1-0) = \cosh 1.\end{aligned}$$

§8.11: 21

Solution: Here we practice more with the delta function, especially the properties noted in Eq. 8.11.19 in Boas. We have

$$\text{(a)} \int_0^3 dx(5x-2)\delta(2-x) = \int_0^3 dx(5x-2)\delta(x-2) = (5\cdot 2-2) = 8[11.19.a],$$

$$\text{(b)} \int_0^\infty dx\phi(x)\delta(x^2-a^2) = \int_0^\infty dx\phi(x)\frac{\delta(x-a)+\delta(x+a)}{|2a|} = \frac{\phi(|a|)}{|2a|}[11.19.d]$$

$$\text{(c)} \int_{-1}^1 dx\cos x\delta(-2x) = \int_{-1}^1 dx\cos x\frac{\delta(x)}{2} = \frac{1}{2}[11.19.c],$$

$$\text{(d)} \int_{-\pi/2}^{\pi/2} dx\cos x\,\delta(\sin x) = \int_{-\pi/2}^{\pi/2} dx\cos x\sum_{n=-\infty}^{\infty}\frac{\delta(x-n\pi)}{|\cos n\pi|} = 1[11.19.e].$$

§8.12: 2

Solution: Let us think about using the Green function technology. Consider the differential equation

$$\ddot{y}+\omega^2 y = \sin\omega t, \left[y(0)=\dot{y}(0)=0\right].$$

We know the corresponding Green function is given by (Eq. 8.12.5)

$$G(t,t') = \begin{cases} 0, 0<t<t' \\ \frac{1}{\omega}\sin\omega(t-t'), 0<t'<t \end{cases}.$$

Thus the solution is given by

$$
\begin{aligned}
y(t) &= \int_0^t dt' \frac{1}{\omega} \sin\omega(t-t') \sin\omega t' \\
&= \frac{1}{-4\omega} \int_0^t dt' \left(e^{i\omega(t-t')} - e^{-i\omega(t-t')}\right)\left(e^{i\omega t'} - e^{-i\omega t'}\right) \\
&= \frac{1}{-4\omega} \int_0^t dt' \left(e^{i\omega t} - e^{-i\omega(t-2t')} - e^{i\omega(t-2t')} + e^{-i\omega t}\right) \\
&= \frac{1}{-4\omega} \left[t'\left(e^{i\omega t} + e^{-i\omega t}\right) - \frac{e^{-i\omega(t-2t')}}{2i\omega} - \frac{e^{i\omega(t-2t')}}{-2i\omega} \right]_0^t \\
&= -\frac{t\cos\omega t}{2\omega} + \frac{\sin\omega t}{2\omega^2} = \frac{\sin\omega t - \omega t\cos\omega t}{2\omega^2}.
\end{aligned}
$$

§8.12: 7

Solution: We start with the differential equation

$$\ddot{y} - a^2 y = e^{-t}, \left[y(0) = \dot{y}(0) = 0 \right].$$

We know the corresponding Green function is given by (exercise 8.12.6)

$$G(t,t') = \begin{cases} 0, 0 < t < t' \\ \frac{1}{a}\sinh a(t-t'), 0 < t' < t \end{cases}.$$

Thus the solution is given by

$$
\begin{aligned}
y(t) &= \int_0^t dt' \frac{1}{a}\sinh a(t-t')e^{-t'} = \frac{1}{2a}\int_0^t dt'\left(e^{a(t-t')}-e^{-a(t-t')}\right)e^{-t'} \\
&= \frac{1}{2a}\int_0^t dt'\left(e^{at-(a+1)t'}-e^{-at+(a-1)t'}\right) = \frac{1}{2a}\left[\frac{e^{at-(a+1)t'}}{-(a+1)}-\frac{e^{-at+(a-1)t'}}{(a-1)}\right]_0^t \\
&= -\frac{1}{2a}\left[\frac{e^{-t}-e^{at}}{(a+1)}+\frac{e^{-t}-e^{-at}}{(a-1)}\right] = -\frac{e^{-t}}{a^2-1}+\frac{\cosh at}{a^2-1}-\frac{\sinh at}{a\left(a^2-1\right)} \\
&= \frac{a\left(\cosh at-e^{-t}\right)-\sinh at}{a\left(a^2-1\right)}.
\end{aligned}
$$

§8.12: 11

<u>Solution</u>: Finally consider the equation

$$
y''+y=\sin 2x, \left[y(0)=y\left(\frac{\pi}{2}\right)=0\right].
$$

We know the corresponding Green function is given by (Eq. 8.12.16)

$$
G(x,x')=\begin{cases}-\cos x'\sin x, & 0<x<x'<\pi/2 \\ -\sin x'\cos x, & 0<x'<x<\pi/2\end{cases}.
$$

Thus the solution is given by (note with these boundary conditions the structure is slightly different from the earlier exercises)

$$
\begin{aligned}
y(x) &= -\cos x\int_0^x dx'\sin x'\sin 2x' - \sin x\int_x^{\pi/2} dx'\cos x'\sin 2x' \\
&= -\cos x\int_0^x dx'\,\frac{\cos(2x'-x')-\cos(2x'+x')}{2} \\
&\quad -\sin x\int_x^{\pi/2} dx'\,\frac{\sin(2x'-x')+\sin(2x'+x')}{2} \\
&= -\cos x\int_0^x dx'\,\frac{\cos(x')-\cos(3x')}{2} - \sin x\int_x^{\pi/2} dx'\,\frac{\sin(x')+\sin(3x')}{2} \\
&= -\frac{\cos x}{2}\left[\sin x' - \frac{\sin 3x'}{3}\right]_0^x + \frac{\sin x}{2}\left[\cos x' + \frac{\cos 3x'}{3}\right]_x^{\pi/2} \\
&= -\frac{\cos x\sin x}{2} + \frac{\cos x\sin 3x}{6} - \frac{\sin x\cos x}{2} - \frac{\sin x\cos 3x}{6} \\
&= -\frac{\sin 2x}{2} + \frac{\sin 2x}{6} = -\frac{\sin 2x}{3}.
\end{aligned}
$$

§8.13: 3

Solution: We want to solve the third order homogeneous differential equation

$$y''' + 2y'' + 2y' = 0.$$

We can turn this into a familiar form with the substitution $z(x) = y'(x)$ so that the equation becomes

$$z'' + 2z' + 2z = 0.$$

This equation we can solve with the usual exponential Ansatz to find

$$z(x) = \beta e^{\alpha x} \Rightarrow \alpha_{1,2} = -\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} = -1 \pm \sqrt{1-2} = -1 \pm i$$
$$\Rightarrow z(x) = e^{-x}\left(A\cos x + B\sin x\right)$$
$$\Rightarrow y(x) = \int dx\, z(x) = e^{-x}\left(C\cos x + D\sin x\right) + E$$

Due to the extra derivative in the original equation there is an extra constant (of integration) term in the solution.

§8.13: 15

<u>Solution</u>: We want to solve the second order inhomogeneous differential equation

$$y'' - 4y' + 4y = 6e^{2x}.$$

To ensure this has well behaved solutions we assume that the driving vanishes for $x < 0$. With this definition we can proceed using Laplace transforms. For now we will also assume boundary conditions $y(0) = y'(0) = 0$. Proceeding to take the transforms we have

$$y'' - 4y' + 4y = 6e^{2x} \Rightarrow \mathfrak{L}[y]\left(p^2 - 4p + 4\right) = \frac{6}{p-2}$$
$$\Rightarrow \mathfrak{L}[y] = \frac{6}{(p-2)^3}.$$

This Laplace transform can be easily inverted using the Table (see $L6$) or by performing the contour integral (where we must expand the exponential to second order to find the residue, $e^{px} = e^{2x+(p-2)x} = e^{2x}\left(1 + (p-2)x + (p-2)^2 x^2/2 + \cdots\right)$ to find

$$y_p(x) = 3x^2 e^{2x}.$$

To match arbitrary boundary conditions we simply add the complementary solution with 2 arbitrary constants to be fitted to the boundary conditions,

$$y(x)=(ax+b)e^{2x}+3x^2e^{2x},$$

the form of which we can read off from the transfer function p^2-4p+4.

§8.13: 30

<u>Solution</u>: We consider the viscous damped motion in gravity described by the equation (the *y* direction is down) with non-constant coefficients

$$m\ddot{y}+\frac{2m}{1+t}\dot{y}=mg\left[y(0)=\dot{y}(0)=0\right].$$

To simplify we can turn this into a first order equation with the substitution $z(t)=\dot{y}(t)$. Thus we have

$$\dot{z}+\frac{2}{1+t}z=g\left[z(0)=0\right].$$

One way to proceed is the old fashioned approach of finding the complementary solution (of the homogeneous equation)

$$\dot{z}_c+\frac{2z_c}{1+t}=0\Rightarrow\frac{dz_c}{z_c}=-\frac{2dt}{1+t}\Rightarrow\ln z_c=-2\ln(1+t)+c$$
$$\Rightarrow z_c(t)=\frac{\overline{c}}{(1+t)^2},$$

and then find a particular solution

$$z_p=a+bt\Rightarrow b+2\frac{(a+bt)}{1+t}=g\Rightarrow a=b=\frac{g}{3}.$$

Adding the two terms and fitting the initial condition yields

$$z(t)=\frac{\bar{c}}{(1+t)^2}+\frac{g}{3}(1+t),\; z(0)=0\Rightarrow \bar{c}=-\frac{g}{3}$$

$$\Rightarrow z(t)=\frac{g}{3}\left(1+t-\frac{1}{(1+t)^2}\right).$$

To find y we integrate again (using the boundary condition)

$$y(t)=\int_0^t dt' z(t')=\frac{g}{3}\left(t+\frac{t^2}{2}+\frac{1}{1+t}-1\right)=\frac{gt}{3}\left(1+\frac{t}{2}-\frac{1}{1+t}\right)$$

$$y(1)=\frac{g}{3}\left(1+\frac{1}{2}-\frac{1}{2}\right)=\frac{g}{3},$$

$$v=\dot{y}(1)=z(1)=\frac{g}{3}\left(1+1-\frac{1}{2^2}\right)=\frac{g}{3}\frac{7}{4}=\frac{7g}{12},$$

$$a=\ddot{y}(1)=\dot{z}(1)=\frac{g}{3}\left(1+\frac{2}{2^3}\right)=\frac{g}{3}\frac{5}{4}=\frac{5g}{12}.$$

§8.13: 47

Solution: : Lets solve $\ddot{y}+y=\sec^2 t=1/\cos^2 t$ via the Green function of 8.12.5 ($\omega=1$) to find

$$y_p(t)=\int_0^t dt'\frac{\sin(t-t')}{\cos^2 t'}=\int_0^t dt'\frac{\sin t\cos t'-\sin t'\cos t}{\cos^2 t'}$$

$$=\sin t\int_0^t\frac{dt'}{\cos t'}-\cos t\int_0^t\frac{\sin t' dt'}{\cos^2 t'}$$

$$=\sin t\ln\left|\tan t'+\sec t'\right|\Big|_0^t-\cos t\left.\frac{1}{\cos t'}\right|_0^t$$

$$=\sin t\ln\left|\tan t+\sec t\right|-1+\cos t.$$

Then the general solution, which can fit more general initial conditions, has the form (*i.e.*, add a the complementary solution)

$$y(t) = A\cos t + B\sin t + \sin t \ln\left|\tan t + \sec t\right| - 1.$$