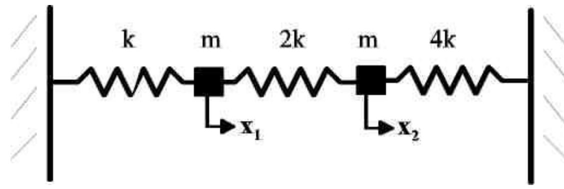
Lecture 20 – Appendix A: Coupled Harmonic Oscillators



As a further example of Lagrangian techniques in classical mechanics let us return to the mass and spring systems we discussed earlier. The system consists of two equal masses suspended between 3 springs as indicated in the figure. The masses are confined to move without friction in 1 dimension. In terms of the indicated coordinates, x_1 and x_2, the total kinetic energy and total potential energy of the system (of 2 masses and 3 springs) are

$$\begin{aligned} T &= \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2\right), \\ V &= \frac{1}{2}k\left(x_1^2 + 4x_2^2 + 2\left\{x_1 - x_2\right\}^2\right) = \frac{1}{2}k\left(3x_1^2 + 6x_2^2 - 4x_1x_2\right), \end{aligned} \tag{A.1}$$

where we take $m = 1$ kg and $k = 1$ N/m = 1 kg/sec^2. The corresponding Lagrangian is thus

$$L\left(x_1, x_2, \dot{x}_1, \dot{x}_2\right) = T - V = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2\right) - \frac{1}{2}k\left(3x_1^2 + 6x_2^2 - 4x_1x_2\right). \tag{A.2}$$

From Lagrange's equations we have the following (coupled) equations of motion,

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} &= m\ddot{x}_1 + 3kx_1 - 2kx_2 = 0, \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} - \frac{\partial L}{\partial x_2} &= m\ddot{x}_2 + 6kx_2 - 2kx_1 = 0. \end{aligned} \tag{A.3}$$

Although these simultaneous equations are coupled, they are still simple enough to solve directly. However, we are encouraged to use the eigenvalue/vector approach to decouple them. In particular, with the usual Ansatz we find the following eigen-equations in matrix form and the corresponding eigenvalues

$$\begin{aligned} &\omega^2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{k}{m}\begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \lambda_1 + \lambda_2 = 9, \lambda_1\lambda_2 = \left(18 - 4\right) = 14 \\ &\Rightarrow \lambda_1 = 2, \lambda_2 = 7. \end{aligned} \tag{A.4}$$

Thus the corresponding eigenfrequencies are (recall the units are radians/sec, and NOT Hz)

$$\omega_1 = \sqrt{\frac{k_1}{m}} = \sqrt{2}\sqrt{\frac{1\text{ Nm}}{1\text{ kg}}} = \frac{\sqrt{2}\text{ rad}}{\text{sec}},$$
$$\omega_2 = \sqrt{\frac{k_2}{m}} = \sqrt{\frac{7\text{ Nm}}{1\text{ kg}}} = \frac{\sqrt{7}\text{ rad}}{\text{sec}}. \tag{A.5}$$

To see the diagonalization we determine the eigenvectors (normal modes) from

$$\lambda_1 : \begin{pmatrix} 3-2 & -2 \\ -2 & 6-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow x_{1,1} - 2x_{2,1} = 0 \Rightarrow \tilde{x}_1 \sim \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix},$$
$$\lambda_2 : \begin{pmatrix} 3-7 & -2 \\ -2 & 6-7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow -4x_{1,2} - 2x_{2,2} = 0 \Rightarrow \tilde{x}_2 \sim \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix}. \tag{A.6}$$

We know that the matrices that diagonalize the potential energy matrix also give us the normal modes and they can be constructed with the above modes as columns,

$$C = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \Rightarrow C^{-1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}. \tag{A.7}$$

As usual starting with a Hermitian matrix we have an orthogonal transformation. The relations between the normal modes and the initial coordinates are

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = C^{-1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2x_1 + x_2 \\ -x_1 + 2x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2\tilde{x}_1 - \tilde{x}_2 \\ \tilde{x}_1 + 2\tilde{x}_2 \end{pmatrix}. \tag{A.8}$$

With the change of coordinates $x_1 \to \left(2\tilde{x}_1 - \tilde{x}_2\right)/\sqrt{5}$ and $x_2 \to \left(\tilde{x}_1 + 2\tilde{x}_2\right)/\sqrt{5}$ (and some arithmetic) the Lagrangian becomes

$$L\left(\tilde{x}_1, \tilde{x}_2, \dot{\tilde{x}}_1, \dot{\tilde{x}}_2\right) = T - V = \frac{1}{2}m\left(\dot{\tilde{x}}_1^2 + \dot{\tilde{x}}_2^2\right) - \frac{1}{2}k\left(2\tilde{x}_1^2 + 7\tilde{x}_2^2\right). \tag{A.9}$$

With no cross terms Lagrange's equations now become

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{\tilde{x}}_1} - \frac{\partial L}{\partial \tilde{x}_1} &= m\ddot{\tilde{x}}_1 + 2k\tilde{x}_1 = 0, \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\tilde{x}}_2} - \frac{\partial L}{\partial \tilde{x}_2} &= m\ddot{\tilde{x}}_2 + 7k\tilde{x}_2 = 0. \end{aligned} \tag{A.10}$$

These are just the equations for 2 *independent* harmonic oscillators. We can obtain the general solutions easily with the (familiar!) complex Ansatz $\tilde{x}(t) = \mathrm{Re}\left[z_0 e^{i\omega t}\right]$, with z_0 a *complex* constant (since this is a linear differential equation). Substituting this Ansatz in complex form where derivatives become just factors of $i\omega$ we have

$$\begin{aligned} m\ddot{\tilde{x}}_1 + 2k\tilde{x}_1 &\Rightarrow \left(-m\omega_1^2 + 2k\right)\left(z_1 e^{i\omega_1 t}\right) = 0 \Rightarrow \omega_1 = \sqrt{\frac{2k}{m}} = \frac{\sqrt{2}}{s}, \\ m\ddot{\tilde{x}}_2 + 7k\tilde{x}_2 &\Rightarrow \left(-m\omega_2^2 + 7k\right)\left(z_2 e^{i\omega_2 t}\right) = 0 \Rightarrow \omega_2 = \sqrt{\frac{7k}{m}} = \frac{\sqrt{7}}{s}. \end{aligned} \tag{A.11}$$

Thus the general time dependence for the normal modes (the simple time dependence due to the decoupled problem) is given by

$$\begin{aligned} \tilde{x}_1(t) &= \mathrm{Re}\left[z_1 e^{i\omega_1 t}\right] = |z_1|\cos(\omega_1 t + \phi_1) : \left[z_1 = |z_1| e^{i\phi_1}\right], \\ \tilde{x}_2(t) &= \mathrm{Re}\left[z_2 e^{i\omega_2 t}\right] = |z_2|\cos(\omega_2 t + \phi_2) : \left[z_2 = |z_2| e^{i\phi_2}\right]. \end{aligned} \tag{A.12}$$

Thus, in terms of the physical coordinates, we have the more complicated time dependence (recall Eq. (A.8))

$$\begin{aligned} x_1(t) &= \frac{1}{\sqrt{5}}\left(2\tilde{x}_1(t) - \tilde{x}_2(t)\right) \\ &= \frac{1}{\sqrt{5}}\left(2|z_1|\cos(\omega_1 t + \phi_1) - |z_2|\cos(\omega_2 t + \phi_2)\right), \\ x_2(t) &= \frac{1}{\sqrt{5}}\left(\tilde{x}_1(t) + 2\tilde{x}_2(t)\right) \\ &= \frac{1}{\sqrt{5}}\left(|z_1|\cos(\omega_1 t + \phi_1) + 2|z_2|\cos(\omega_2 t + \phi_2)\right). \end{aligned} \tag{A.13}$$

To find an example complete solution, we apply the initial conditions from the exam,

$$x_1(0) = 1\text{ cm},\ \dot{x}_1(0) = 0,$$
$$x_2(0) = 0,\ \dot{x}_2(0) = 0. \tag{A.14}$$

These initial conditions for the physical coordinates turn into the following initial conditions for the normal coordinates (again using Eq. (A.8))

$$\begin{aligned}
\tilde{x}_1(0) &= \frac{1}{\sqrt{5}}\left(2x_1(0) + x_2(0)\right) = \frac{2\text{ cm}}{\sqrt{5}},\\
\tilde{x}_2(0) &= \frac{1}{\sqrt{5}}\left(-x_1(0) + 2x_2(0)\right) = -\frac{1\text{ cm}}{\sqrt{5}},\\
\dot{\tilde{x}}_1(0) &= \dot{\tilde{x}}_2(0) = \frac{1}{\sqrt{5}}\left(\dot{x}_1(0) \mp \dot{x}_2(0)\right) = 0\\
&\Rightarrow |z_1| = \frac{1\text{ cm}}{\sqrt{5}}, |z_2| = \frac{2\text{ cm}}{\sqrt{5}}, \phi_1 = \phi_2 = 0.
\end{aligned} \tag{A.15}$$

Thus the time dependence of the normal modes looks like

$$\begin{aligned}
\tilde{x}_1(t) &= \frac{2\text{ cm}}{\sqrt{5}}\cos\left(\sqrt{2}\frac{t}{\text{sec}}\right),\\
\tilde{x}_2(t) &= -\frac{1\text{ cm}}{\sqrt{5}}\cos\left(\sqrt{7}\frac{t}{\text{sec}}\right).
\end{aligned} \tag{A.16}$$

Finally we transform back to the physical coordinates and find the desired result

$$\begin{aligned}
x_1(t) &= \frac{1}{\sqrt{5}}\left(2\tilde{x}_1(t) - \tilde{x}_2(t)\right) = \frac{1\text{ cm}}{5}\left(4\cos\left(\sqrt{2}\frac{t}{\text{sec}}\right) + \cos\left(\sqrt{7}\frac{t}{\text{sec}}\right)\right),\\
x_2(t) &= \frac{1}{\sqrt{5}}\left(\tilde{x}_1(t) + 2\tilde{x}_2(t)\right) = \frac{2\text{ cm}}{5}\left(\cos\left(\sqrt{2}\frac{t}{\text{sec}}\right) - \cos\left(\sqrt{7}\frac{t}{\text{sec}}\right)\right).
\end{aligned} \tag{A.17}$$

Note that at $t = 0$, $\ddot{x}_1(0) < 0$ and $\ddot{x}_2(0) > 0$ as we expect on physical grounds (the middle spring is initially compressed and pushes to the right on m_2, while both the left-hand spring and the middle spring are applying leftward forces on m_1).