Lecture 7 – Appendix B: Some sample problems from Chapter 7 in K & B. As you think about these exercises, you are encouraged to use *Mathematica* (or other similar programs) to evaluate the various quantities numerically and make plots (like any here).

Exercise 7.4 in K & B

<u>Solution</u>: We want to consider the motion of a system composed of 2 particles attached by a spring of natural length *l* and spring constant *k*. Both masses also experience a uniform (downward) gravitational force. In this case of uniform external potential the motion completely factors into the CM motion and the relative motion as in Eq. (7.10). We have a Lagrangian and initial conditions

$$
\begin{aligned}
L = T - V &= \frac{1}{2}\left(m_1\dot{\vec{r}}_2^{\,2} + m_2\dot{\vec{r}}_2^{\,2}\right) + m_1\vec{g}\cdot\vec{r}_1 + m_2\vec{g}\cdot\vec{r}_2 - V_{\text{int}}\left(\vec{r}_1 - \vec{r}_2\right) \\
&= \frac{1}{2}\left(m_1\dot{\vec{r}}_2^{\,2} + m_2\dot{\vec{r}}_2^{\,2}\right) - m_1 g z_1 - m_2 g z_2 - \frac{k}{2}\left(\left|\vec{r}_1 - \vec{r}_2\right| - l\right)^2 \\
&= \frac{1}{2}\left(M\dot{\vec{R}}^2 + \mu\dot{\vec{r}}^{\,2}\right) - Mg\hat{z}\cdot\vec{R} - \frac{k}{2}\left(r - l\right)^2,
\end{aligned}
$$

$$
\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}\,\left[\vec{r}_1\left(0\right) = l\hat{z},\, \dot{\vec{r}}_1\left(0\right) = v\hat{z}\right],
$$

$$
\vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}\,\left[\vec{r}_2\left(0\right) = \vec{0},\, \dot{\vec{r}}_2\left(0\right) = \vec{0}\right],
$$

$$
\Rightarrow \vec{R}\left(0\right) = \frac{m_1}{M} l\hat{z},\, \dot{\vec{R}}\left(0\right) = \frac{m_1}{M} v\hat{z},\, \vec{r}\left(0\right) = l\hat{z},\, \dot{\vec{r}}\left(0\right) = v\hat{z}.
$$

With the given boundary conditions the motion will only be in the z direction and we have equations of motion for the separated motions

$$
\ddot{Z} = -g \Rightarrow Z\left(t\right) = \frac{m_1}{M} l + \frac{m_1}{M} vt - \frac{g}{2}t^2,
$$

$$
\ddot{z} = -\frac{k}{\mu}\left(z - l\right) \Rightarrow z\left(t\right) = l + v\sqrt{\frac{\mu}{k}}\sin\left(\sqrt{\frac{k}{\mu}}t\right),
$$

where we have used our prior experience with these equations and the given initial conditions to write down the answers. Thus the locations of the 2 masses are

$$\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}$$

$$\Rightarrow z_1(t) = Z(t) + \frac{m_2}{M} z(t) = l + \frac{m_1}{M} vt - \frac{g}{2}t^2 + \frac{m_2}{M} v\sqrt{\frac{\mu}{k}} \sin\left(\sqrt{\frac{k}{\mu}}t\right),$$

$$\vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$$

$$\Rightarrow z_2(t) = Z(t) - \frac{m_1}{M} z(t) = \frac{m_1}{M} vt - \frac{g}{2}t^2 - \frac{m_1}{M} v\sqrt{\frac{\mu}{k}} \sin\left(\sqrt{\frac{k}{\mu}}t\right).$$

The constraint that the spring not be overly compressed is that the relative coordinate (separation) z never become negative. This requires that $l > v\sqrt{m/k}$.

Exercise 7.5 in K&B

Solution: Let's think some about elastic scattering and the results on scatterings from the Lecture. Recall that the angle of the recoil particle (the one at rest in the lab frame) is given by Eq. (7.21) and (7.22) (using some helpful trig identities and writing everything in terms of the CM angle over 2)

$$\alpha = \frac{\pi - \theta^*}{2} \Rightarrow \tan\alpha = \tan\left(\frac{\pi - \theta^*}{2}\right) = \cot\left(\frac{\theta^*}{2}\right) = \frac{1}{\tan\left(\theta^*/2\right)},$$

$$\tan\theta = \frac{\sin\theta^*}{\left(\frac{m_1}{m_2} + \cos\theta^*\right)} = \frac{2\sin\left(\theta^*/2\right)\cos\left(\theta^*/2\right)}{\frac{m_1}{m_2}\left(\sin^2\left(\theta^*/2\right) + \cos^2\left(\theta^*/2\right)\right) + \cos^2\left(\theta^*/2\right) - \sin^2\left(\theta^*/2\right)}$$

$$= \frac{2\tan\left(\theta^*/2\right)}{\frac{m_1}{m_2}\left(1 + \tan^2\left(\theta^*/2\right)\right) + 1 - \tan^2\left(\theta^*/2\right)}$$

$$\Rightarrow \tan\alpha + \tan\theta = \frac{\left(1 + \tan^2\left(\theta^*/2\right)\right)\left(1 + m_1/m_2\right)}{\tan\left(\theta^*/2\right)\left[\left(1 - \tan^2\left(\theta^*/2\right)\right) + m_1/m_2\left(1 + \tan^2\left(\theta^*/2\right)\right)\right]},$$

$$\tan\alpha\tan\theta = \frac{2}{\left(1 - \tan^2\left(\theta^*/2\right)\right) + m_1/m_2\left(1 + \tan^2\left(\theta^*/2\right)\right)}.$$

Using more trig identities we can simplify the desired ratio via

$$\tan(\alpha+\theta) = \frac{\sin(\alpha+\theta)}{\cos(\alpha+\theta)} = \frac{\sin\alpha\cos\theta + \sin\theta\cos\alpha}{\cos\alpha\cos\theta - \sin\alpha\sin\theta} = \frac{\tan\alpha + \tan\theta}{1-\tan\alpha\tan\theta}$$

$$= \frac{\left(1+\tan^2\left(\theta^*/2\right)\right)\left(1+m_1/m_2\right)}{\tan\left(\theta^*/2\right)\left[\left(1+\tan^2\left(\theta^*/2\right)\right)\left(1-m_1/m_2\right)\right]} = \frac{\left(1+m_1/m_2\right)}{\tan\left(\theta^*/2\right)\left(1-m_1/m_2\right)}$$

$$\Rightarrow \frac{\tan(\alpha+\theta)}{\tan(\alpha)} = \frac{\left(1+m_1/m_2\right)}{\left(1-m_1/m_2\right)} = \frac{m_1+m_2}{m_1-m_2}.$$

Thus we note that in order for the two scattered particles to come out at right angles in the lab, $\alpha+\theta=\pi/2$, both the LHS and the RHS of this equation must go to infinity. For the RHS this can only arise if $m_1 = m_2$.

Exercise 7.6 in K&B

<u>Solution</u>: Consider another application of the elastic scattering problem discussed in the Lecture in the laboratory frame (particle 2 initially at rest). In Eq. (7.22) we found that, for a CM scattering angle θ^*, the incoming particle is scattered through an angle given by

$$\tan\theta = \frac{\sqrt{1-\cos^2\theta}}{\cos\theta} = \frac{\sqrt{1-\cos^2\theta^*}}{\left(\frac{m_1}{m_2}+\cos\theta^*\right)} = \frac{\sin\theta^*}{\left(\frac{m_1}{m_2}+\cos\theta^*\right)},$$

while the recoiling particle appears at an angle (see Eq. (7.21))

$$\cos\alpha = \sin\frac{\theta^*}{2}.$$

Thus for the specified conditions we

$$\sin\frac{\theta^*}{2} = \cos\alpha = \cos 60° = \frac{1}{2} \Rightarrow \theta^* = 2\sin^{-1}\left(\cos 60°\right) = 60°$$

$$\Rightarrow \sin\theta^* = \frac{\sqrt{3}}{2}, \cos\theta^* = \frac{1}{2}$$

$$\Rightarrow \frac{m_2}{m_1} = \left(\frac{\sin\theta^*}{\tan\theta} - \cos\theta^*\right)^{-1} = \left(\frac{\sin 60°}{\tan 56°} - \cos 60°\right)^{-1} = \left(0.0841\right)^{-1} = 11.9.$$

So the atomic number of the target is 12. The energy transferred is given by (see Eq. (7.25))

$$\frac{T_2}{T} = \frac{4m_1m_2}{M^2}\sin^2\frac{\theta^*}{2} = \frac{4\times1\times12}{13^2}\left(\frac{1}{2}\right)^2 = \frac{12}{13^2} = 0.071.$$