1. Consider a sphere, of radius $R$ and uniform charge density such that the total charge is $Q$.

(a) (5) There is a second Gaussian sphere of radius $r > R$ with a center which is a distance $2r$ away from the center of the charge carrying sphere. Determine the electric flux through the Gaussian sphere.

(b) (5) Now suppose the Gaussian sphere of radius $r > R$ has the same center as the original sphere. Determine the electric flux through the second sphere for this new situation.

(c) (10) Consider again the Gaussian sphere of part (b). Show that the surface integral over the Gaussian sphere is given by $\oint \mathbf{E}(r) \cdot \hat{n} dA = 4\pi r^2 E(r)$.

(d) (10) Consider $r < R$, show that $E(r) = \frac{1}{4\pi \varepsilon_0} \frac{Q}{r^2}$. 

\[ E(r) = \frac{1}{4\pi \varepsilon_0} \int_0^r \frac{Q}{4\pi r^2} dr \]

\[ E(r) = \frac{1}{4\pi \varepsilon_0} \frac{Q}{3} \]

\[ E(r) = \frac{1}{4\pi \varepsilon_0} \frac{Q}{r^2} \]
(e) (15) Determine the electrostatic energy stored in the sphere of radius $R$ and charge $Q$.

$$W = \frac{1}{2} \varepsilon_0 \int d^3r \ \vec{E}^2$$

$$= \frac{\varepsilon_0}{2} \left\{ \int_0^R r^2 dr \int_0^\infty \frac{E^2}{r} dr + \int_0^\infty r^2 dr \int_0^\infty \frac{E^2}{r} dr \right\}$$

$$= \varepsilon_0 2\pi \left[ \int_0^R r^2 dr \left( \frac{Q^2}{4\pi \varepsilon_0 R^2} \right) + \int_0^\infty r^2 dr \left( \frac{Q^2}{4\pi \varepsilon_0 r^2} \right) \right]$$

$$= \frac{\varepsilon_0 2\pi}{16\pi^2 \varepsilon_0} \frac{Q^2}{R^2} \left[ \int_0^R \frac{r^2 dr}{R^6} + \int_0^\infty \frac{r^2 dr}{r^4} \right]$$

$$= \frac{Q^2}{8 \pi \varepsilon_0} \left[ \frac{R^5}{5R^6} + \frac{1}{R} \right] = \frac{Q^2}{4\pi \varepsilon_0 R} \left( \frac{3}{5} \right)$$
2. This problem is concerned with a thin disk of radius \( R \) with a charge per unit area of \( \sigma \).

(a) (10) Compute the electric potential \( V(z) \) at a distance \( z \) on the axis of the disk (directly above the center of the disk).

\[
V(z) = \frac{1}{4\pi \varepsilon_0} \int \frac{dQ \cdot \sigma}{r} = \frac{1}{4\pi \varepsilon_0} \sigma \int_0^R \int_{-\pi}^{\pi} \frac{d\sigma}{\sqrt{z^2 + s^2}}
\]

(b) (10) Determine the electric field \( \vec{E}(z) \) at that same point.

\[
\vec{E}(z) = \vec{E}(z) z \quad (\text{from cylindrical symmetry})
\]

\[
\vec{E}(z) = -\frac{\partial V}{\partial z} = \frac{1}{2\varepsilon_0} \left[ \frac{z}{\sqrt{R^2 + z^2}} \right] - \vec{J}
\]

(c) (10) Suppose \( R \gg z \), what should the electric field \( \vec{E}(z) \) be?

\[
R \gg z \quad E = \frac{\sigma}{2\varepsilon_0}
\]

from above, or by application of Gauss Law.
(d) (15) Now consider positions close to the axis, but a distance $s \ll R_0^2$ away from the axis. Obtain the potential $V(s, d)$. Your answer should be expressed in terms of a well-defined double integral. Do not carry out the integral.

$$V(s, d) = \frac{1}{4 \pi \varepsilon_0} \int_0^{2\pi} \int_0^R \frac{ds'dd'}{d'} \frac{1}{|r - r'|}$$

$$|r - r'| = \sqrt{z^2 + (x-x')^2 + (y-y')^2} = \sqrt{z^2 + s^2 + s'^2 - 2ss' \cos \phi}$$

$$V(s, d) = \frac{1}{4 \pi \varepsilon_0} \int_0^{2\pi} \int_0^R \frac{d\phi}{\sqrt{z^2 + s^2 + s'^2 - 2ss' \cos \phi}}$$

(e) (10) Explain why $V(s, z)$ is well approximated by the expression of the form $V(s, z) = V(0, z) + s^2 f(z)$. 

Thus any dependence on $s^2$ must lead to

$S^4$ and higher powers are ignored because $S^2$ is small.

$$E_x = E_y = 0 \quad \text{for} \quad x = y = 0$$

Cylindrical symmetry means $V(0, z)$

$S^2 = x^2 + y^2$

$E_x \partial^2 x = 0 \quad \text{for} \quad x = 0$

$E_y \partial^2 y = 0 \quad \text{for} \quad y = 0$