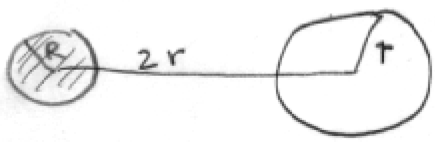
321 midterm 1 October 2000

1. Consider a sphere, of radius R and uniform charge density such that the total charge is Q.

(a) (5) There is a second Gaussian sphere of radius $r > R$ with a center which is a distance $2r$ away from the center of the charge carrying sphere. Determine the electric flux through the Gaussian sphere.



No charge enclosed

elec flux $= 0$

(b) (5) Now suppose the Gaussian sphere of radius $r > R$ has the same center as the original sphere. Determine the electric flux through the second sphere for this new situation.

Electric flux $= Q/\epsilon_0$

(c) (10) Consider again the Gaussian sphere of part (b). Show that the surface integral over the Gaussian sphere is given by $\oint \vec{E}(\vec{r}) \cdot \hat{n} dA = 4\pi r^2 E(r)$.

Spherically symmetric charge distribution

$\vec{E}(\vec{r}) = E(r)\hat{r}$. Sphere $\hat{n} = \hat{r}$

$\vec{E}\cdot\hat{n} = E(r)$

$$\oint_{\text{Sphere}} \vec{E}\cdot\hat{n}\, da = \int_{\text{Sphere}} E(r)\, dA = E(r)\int dA = E(r)\, 4\pi r^2$$

(d) (10) Consider $r < R$, show that $E(r) = \frac{1}{\cancel{3}\epsilon_0} \frac{\cancel{3}Q}{4\pi R^3}\, r$.

$E 4\pi r^2$ = charge enclosed $/ \epsilon_0$ $\qquad \rho = \frac{Q3}{4\pi R^3}$

$$= \frac{1}{\epsilon_0}\int_0^r 4\pi r^2 dr \frac{Q\,3}{4\pi R^3}$$

$$E\cancel{4\pi} r^2 = \frac{1}{\epsilon_0}\cancel{4\pi}\,\frac{r^3}{3}\,\frac{3Q}{4\pi R^3}$$

$$\therefore E = \frac{1}{\epsilon_0}\,\frac{r^2\, Q}{4\pi R^3}$$

(e) (15) Determine the electrostatic energy stored in the sphere of radius R and charge Q.

$$W = \frac{1}{2}\epsilon_0 \int d^3r \; E^2$$

$$= \frac{\epsilon_0}{2} 4\pi \left\{ \int_0^R r^2 dr\, E^2 + \int_R^\infty r^2 dr\, E^2 \right\}$$

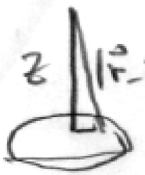
$$= \epsilon_0 2\pi \left[\int_0^R r^2 dr \left(\frac{r\, Q}{4\pi R^3 \epsilon_0}\right)^2 + \int_R^\infty r^2 dr \left(\frac{Q}{4\pi \epsilon_0 r^2}\right)^2 \right]$$

$$= \frac{\epsilon_0 2\pi}{16\pi^2 \epsilon_0^2} Q^2 \left[\int_0^R \frac{r^2 dr\, r^2}{R^6} + \int_R^\infty \frac{r^2 dr}{r^4} \right]$$

$$= \frac{Q^2}{8\pi \epsilon_0} \left[\frac{R^5}{5R^6} + \frac{1}{R} \right] = \frac{Q^2}{4\pi \epsilon_0 R} \frac{3}{5}$$

2. This problem is concerned with a thin disk of radius R with a charge per unit area of σ.

(a) (10) Compute the electric potential $V(z)$ at a distance z on the axis of the disk (directly above the center of the disk).



$$V(P) = \frac{1}{4\pi\epsilon_0}\int \frac{da'\,\sigma}{|\vec{r}-\vec{r}'|} = \frac{1}{4\pi\epsilon_0}\,\sigma\int_0^R s\,ds\int_0^{2\pi}\frac{d\phi}{\sqrt{z^2+s^2}}$$

$dq = \sigma\, dA$

$\int \frac{da'\,\sigma}{\mathscr{r}}$

$$= \frac{1}{4\pi\epsilon_0}\,\sigma\, 2\pi \int_0^R \frac{s\,ds}{\sqrt{s^2+z^2}} = \frac{1}{4\pi\epsilon_0}\,2\pi\sigma\left(\sqrt{R^2+z^2}-z\right)$$

$da = s\,ds\,d\phi$

(b) (10) Determine the electric field $\vec{E}(z)$ at that same point.

$\vec{E} = E(z)\,\hat{z}$ (from cylindrical symmetry)

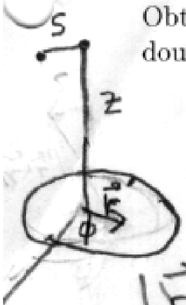
$$E(z) = -\frac{\partial V}{\partial z} = \frac{1}{2}\frac{\sigma}{\epsilon_0}\left[\frac{z}{\sqrt{R^2+z^2}} - 1\right]$$

(c) (10) Suppose $R \gg z$, what should the electric field $\vec{E}(z)$ be?

$R \gg z$

$E = \frac{\sigma}{2\epsilon_0}$ from above, or by application of Gauss Law

(d) (15) Now consider positions close to the axis, but a distance $s \ll R, z$ away from the axis. Obtain the potential $V(s, d)$. Your answer should be expressed in terms of a well-defined double integral. Do not carry out the integral.



$$V(s,d) = \frac{1}{4\pi\varepsilon_0}\,\sigma \int_0^{R} s'ds' \int_0^{2\pi} d\phi \, \frac{1}{|\vec{r}-\vec{r}\,'|}$$

$$|\vec{r}-\vec{r}\,'| = \sqrt{z^2 + (x-x')^2 + (y-y')^2} = \sqrt{z^2+s^2+s'^2-2ss'\cos\phi}$$

$$V(s,d) = \frac{1}{4\pi\varepsilon_0}\,\sigma \int_0^{R} s'ds' \int_0^{2\pi} \frac{d\phi}{\sqrt{z^2+s^2+s'^2-2ss'\cos\phi}}$$

(e) (10) Explain why $V(s, z)$ (for $s \ll R, z$) is well approximated by the expression of the form $V(s, z) = V(0, z) + s^2 f(z)$.

$E_x = E_y = 0$ for $x = y = 0$, Cylindrical symmetry means $V(s, z)$ depends on s^2

Thus any dependence on s^2 must lead to a vanishing E_x, E_y.

$s^2 = x^2 + y^2$

$E_x \propto 2x = 0$ for $x = 0$

$E_y \propto 2y = 0$ for $y = 0$

s^4 and higher powers are ignored because s^2 is small.