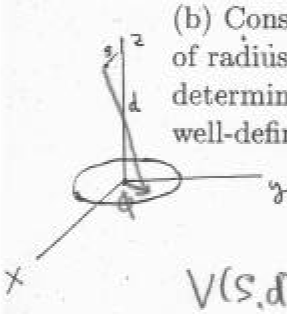
2000 321 Final Exam Solutions

1 (a) (10) For values of $r \neq 0$, state the value of $\nabla^2 \frac{1}{r}$. No derivation is needed.

$$0$$

(b) Consider the electric potential V at the point $x = s, z = d$ which arises from a thin disk of radius R with a constant charge per unit area of σ. See the figure. Suppose that $d \to \infty$, determine the (non-vanishing) value of V (10). Obtain the potential $V(s, d)$ in terms of a well-defined double integral (15).



$$d \to \infty \quad V \to \frac{Q}{4\pi\varepsilon_0 d} = \boxed{\frac{\sigma \pi R^2}{4\pi\varepsilon_0 d}}$$

$$V(s,d) = \frac{1}{4\pi\varepsilon_0} \int \frac{\sigma da'}{|\vec{r}-\vec{r}'|} = \frac{\sigma}{4\pi\varepsilon_0} \int_0^{2\pi} d\phi \int_0^R \frac{s' ds'}{\sqrt{d^2+s^2+s'^2-2ss'\cos\phi'}}$$

$$|\vec{r}-\vec{r}'| = \sqrt{r^2+r'^2-2\vec{r}\cdot\vec{r}'} = \sqrt{d^2+s^2+s'^2-2ss'\cos\phi'}$$

(c) (10) Consider two concentric spheres one of radius a and the other of radius b, $(b > a)$. We are concerned with finding the potential in the region between the two spheres: $a < r < b$. The sphere at $r = a$ is grounded, but the sphere at $r = b$ is held at a potential $V_0 \cos\theta$. Determine the potential $V(r, \theta)$ for $a < r < b$.

$$V = \left(A r + \frac{B}{r^2}\right)\cos\theta$$

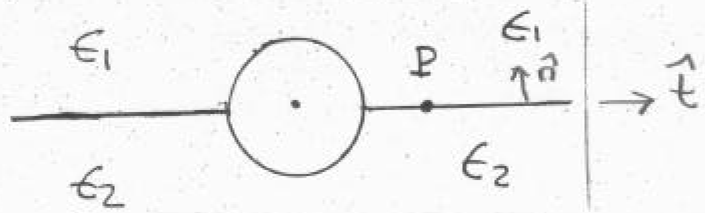
$$V(r=a) = 0 = Aa + \frac{B}{a^2} \qquad B = -Aa^3$$

$$V(r=b) = V_0 \cos\theta = \left(A b + \frac{B}{b^2}\right)\cos\theta$$

$$V_0 = A\left[b - \frac{a^3}{b^2}\right] \qquad A = \frac{b^2 V_0}{b^3-a^3}$$

$$V = V_0 \left[\frac{b^2}{b^3-a^3}\right]\left[r - \frac{a^3}{r^2}\right]\cos\theta$$

2.



The figure shows a conducting sphere, of radius R and free charge Q. The center of the sphere lies on the plane boundary between two infinite linear dielectrics with dielectric constants ϵ_1 and ϵ_2.

(a) (10) Consider the point P, which is on the boundary. What quantities are continuous at the point P?

Normal $\hat{n}$ components of $\vec{D}$
tangential $\hat{t}$ component of $\vec{E}$

(b) (10) Show or explain that the electric potential V takes the form $V = \frac{K}{r}$, where $r \geq R$ is the distance from the center of the sphere and K is a constant to be determined in part (b).

The free charge moves along ~~over~~ the surface to produce a total charge distribution which minimizes the effects of repulsion. This is a spherically symmetric total charge distribution which produces a spherically symmetric potential $V = K/r$

(c) (15) Determine $\vec{E}$ and also the electric displacement $\vec{D}$ (which requires determining K) for positions on and outside the sphere.

$\vec{E} = -\vec{\nabla} V = \boxed{\frac{K\hat{r}}{r^2}}$ above the plane $\vec{D} = \epsilon_1 \vec{E} = \boxed{\frac{\epsilon_1 K \hat{r}}{r^2}}$

below the plane $\vec{D} = \epsilon_2 \vec{E} = \boxed{\frac{\epsilon_2 K \hat{r}}{r^2}}$

at the surface of the sphere $\int \vec{D} \cdot d\vec{a} = Q$

$= 2\pi R^2 \left(\frac{\epsilon_1 K}{R^2} + \frac{\epsilon_2 K}{R^2} \right) = Q$ $\boxed{K = \frac{Q}{2\pi[\epsilon_1 + \epsilon_2]}}$

(d) (15) Find the free and bound surface charge densities on the sphere.

above the plane $\sigma_b = -\vec{P}_1 \cdot \hat{r} = -\epsilon_0 \chi_e \vec{E} \cdot \hat{r} = -\epsilon_0 \chi_e^{(1)} K/R^2$

$\chi_e^{(1)} = \frac{\epsilon_1 - \epsilon_0}{\epsilon_0}$

$\sigma_{free} = \vec{D}_1 \cdot \hat{r} = \epsilon_1 K/R^2$

below the plane $\sigma_b = -\vec{P}_2 \cdot \hat{r} = -\epsilon_0 \chi_e^{(2)} K/R^2$

$\sigma_{free} = \epsilon_2 K/R^2$ $\chi_e^{(2)} = \frac{\epsilon_2 - \epsilon_0}{\epsilon_0}$

CHECK

$\sigma_b + \sigma_f = K/R^2 \left[-\epsilon_0 \chi_e^{(1)} + \epsilon_1 \right] = \epsilon_0 K/R^2$

$\sigma_b + \sigma_f = K/R^2 \left[-\epsilon_0 \chi_e^{(2)} + \epsilon_2 \right] = \epsilon_0 K/R^2$ } same

3.(a) (10) For positions outside an azimuthally symmetric charge distribution the potential takes the form $V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta)$. Explain the meaning of the terms B_0 and B_1.

B_0 is the total charge

B_1 is the z component of the dipole moment

(b) (10) You are given the expression $\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{l=0}^{\infty}\frac{(r')^l}{r^{l+1}}P_l(\widehat{r}\cdot\widehat{r'}) \quad r > r'$. (1) Suppose $\vec{r}$ and $\vec{r}\,'$ are parallel, and $r = 3R$, $r' = R$, with R given. Use (1) to show that $\frac{1}{2} = \sum_{l=0}^{\infty}\left(\frac{1}{3}\right)^{l+1}$.

If $\vec{r} \parallel \vec{r}\,'$ $P_l(\hat{r}\cdot\hat{r}') = 1$ plug in $r = 3R$, $r' = R$ into formula (1)

$$\frac{1}{|3R-R|} = \sum_{l=0}^{\infty}\frac{R^l}{(3R)^{l+1}}$$

$$\frac{1}{2R} = \sum_{l=0}^{\infty}\frac{1}{3^{l+1}}\frac{1}{R} \Rightarrow \frac{1}{2} = \sum_{l=0}^{\infty}\frac{1}{3^{l+1}}$$

(c) (10) A charge distribution has a charge density $\rho(\vec{r})$. Given this density, show that $V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r'\rho(\vec{r'})\sum_{l=0}^{\infty}\frac{(r')^l}{r^{l+1}}P_l(\widehat{r}\cdot\widehat{r'})$. and for values of r such that there is no charge.

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r'\frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} = \frac{1}{4\pi\epsilon_0}\int d^3r'\rho(\vec{r}\,')\sum_{l=0}^{\infty}\frac{r'^l}{r^{l+1}}P_l(\hat{r}\cdot\hat{r}')$$

Substitute (1) valid for $r > r'$ allways

3. (d) (10) The density $\rho(\vec{r})$ is azimuthally symmetric. Determine the coefficients B_l in terms of integrals involving $\rho(\vec{r})$.

Evaluate previous expression at $\theta = 0$. Then $P_\ell(\hat{r}\cdot\hat{r}') = P_\ell(\cos\theta')$

$$V(r,\theta=0) = \frac{1}{4\pi\varepsilon_0} \sum_\ell \frac{\int d^3r' \rho(r')\, r'^\ell P_\ell(\cos\theta')}{r^{\ell+1}}$$

Thus $B_\ell = \int d^3r' \rho(\vec{r}')\, r'^\ell P_\ell(\cos\theta')$

$\rho(\vec{r})$

3. (e) (15) The charge density, $\rho(\vec{r})$, of certain atomic nuclei can be well approximated by $\rho(\vec{r}) = Q/(\pi^{3/2} R^3)\ e^{-r^2/R^2}\left(1 + \beta_2 r^2/R^2\, P_2(\cos\theta)\right)$, where Q is the charge of the nucleus, R is a size parameter ($\sim 4\times10^{-15}$m), and β_2 is a given dimensionless coefficient. There are positions r large enough so that $\rho(\vec{r})$ can be taken to vanish. For such positions, determine $V(r,\theta)$. You are given $\int_0^\infty dx\ e^{-ax^2} = \sqrt{\pi}/2\ a^{-1/2}$, $\int_0^\infty dx\ x^{2n}e^{-ax^2} = \sqrt{\pi}/2 \frac{1\cdot3\cdots(2n-1)}{2^n}\ a^{-(n+1)/2}$ $(n \geq 1)$.

all $B_\ell = 0$ unless $\ell = 0, 2$

$$B_0 = \int d^3r' \frac{Q}{\pi^{3/2}R^3} e^{-r'^2/R^2}\left(1+\beta_2 r'^2/R^2 P_2(\cos\theta')\right)$$

$$= \int d^3r' \frac{Q}{\pi^{3/2}R^3} e^{-r'^2/R^2} = \text{total charge} = Q. \quad \text{do integral to check}$$

$$= 4\pi \int r'^2 dr' \frac{Q}{\pi^{3/2}R^3} e^{-r'^2/R^2} = \frac{4\pi Q}{\pi^{3/2}R^3}\frac{\sqrt{\pi}}{2}\frac{1}{2}R^3 = Q$$

$$B_2 = \int d^3r' \frac{Q}{4\pi^{3/2}R^3} e^{-r'^2/R^2}\left(1+\frac{\beta_2 r'^2}{R^2}P_2(\cos\theta')\right) r'^2 P_2(\cos\theta')$$

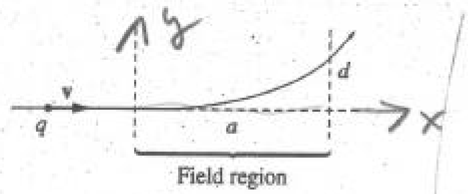
only non-zero

$\int_{-1}^{1} dx\, P_2^2 = \frac{2}{5}$

$$= \frac{Q}{\pi^{3/2}R^3}\frac{\beta_2}{R^2} 2\pi \frac{2}{5}\int dr'\, r'^4 e^{-r'^2/R^2}$$

$$= \frac{Q\beta_2}{5}\ \frac{\ }{2}\frac{4}{2}\frac{3}{2} = \frac{Q\beta_2 3}{10}. \quad V(r,\theta) = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r} + \frac{3\beta_2}{10}\frac{P_2(\cos\theta)}{r^3}\right]$$

4.



(a) A particle of charge q enters a region of uniform magnetic field $\vec{B}$ pointing **out** of the page. The field deflects the particle a distance d above the original line of flight, as shown in the figure.(10) Is the charge positive or negative? (10) Is the speed of the particle constant during its motion in the magnetic field? Explain why or why not. (10) Find the momentum of the pa rticle (as it leaves the field region) in terms of a, d, q and B.

$\vec{v} \times \vec{B}$ is down Force is up. q is negative

The speed is constant because the magnetic force does no work.

The particle moves along a circle of radius R with center at $y = R$

$|q| vB = mv^2/R \qquad mv = |q| B R$, need to find R

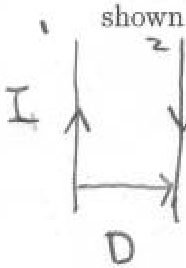
The path is on the point $x = a, y = d$

$(x^2) + (y-R)^2 = R^2 = a^2 + (d-R)^2 = R^2$

$a^2 + d^2 - 2dR = 0 \qquad R = (a^2+d^2)/2d$

momentum $= mv = \boxed{|q| B \; (a^2+d^2)/2d}$

(b) Two very long straight parallel wires separated by a distance $2D$ carry a current I in opposite directions.(10) Is the magnetic force between the two wires attractive or repulsive? Explain. (10) Determine the magnitude and direction of the magnetic field at the point P shown in the figure.



$\vec{B}_1$ caused by wire 1 is <u>into</u> page at position of wire 2

$F_{on\ 2}$ is $\parallel (\vec{I}_2 \times \vec{B})$ $\qquad \vec{I}_2 \times \vec{B}_1$ to the right

$\boxed{\text{repulsive}}$

B is sum of B's caused by two wires

$$|B| = \frac{\mu_0 I}{2\pi}\left[\frac{1}{4D} - \frac{1}{6D}\right] = \frac{\mu_0 I}{2\pi} \frac{1}{12D}$$

$\vec{B}$ points <u>out</u> of the page