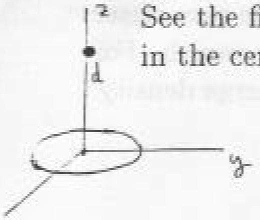
321 Final exam 2001 Solutions

1 (a) (15) Consider a **ring** of charge of radius R with a constant charge per unit length, λ. See the figure. Determine the charge and dipole moment of the ring, taking the origin to be in the center of the ring. Does the quadrupole moment of the ring vanish?



Charge $= 2\pi R \lambda$

Dipole moment $= 0$ because $\int \vec{r}\,\lambda\, dl = 0$

Quadrupole moment does not vanish because we expect the term $\frac{B_2 P_2}{r^3}$ to be present (also $\frac{B_\ell P_\ell}{r^3}$ with $\ell =$ odd vanish)

(b) (10) Determine the potential at a point which is on the axis, a finite distance d away from the center.

$$V = \frac{Q}{4\pi\varepsilon_0} \frac{1}{\sqrt{R^2+d^2}}$$

because all points on the ring are $\sqrt{R^2+d^2}$ away from the point at d

(c)(25) Now determine the potential $V(\vec{r}\,)$, in terms of an expansion in Legendre polynomials, at a point where $r > R$. Hint: $(1+x)^n = 1 + \frac{n}{1!}\,x + \frac{n(n-1)}{2!}\,x^2 + \frac{n(n-1)(n-2)}{3!}\,x^3 + \cdots$.

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \; ; \; V(r,0) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} = \frac{Q}{4\pi\varepsilon_0\sqrt{R^2+r^2}}$$

$$\frac{1}{\sqrt{R^2+r^2}} = \frac{1}{r}\frac{1}{\sqrt{1+\frac{R^2}{r^2}}} = \frac{1}{r}\left(1 - \frac{1}{2}\frac{R^2}{r^2} + \frac{3}{2}\frac{1}{2}\frac{1}{2}\left(\frac{R^2}{r^2}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{R^2}{r^2}\right)^3 + \cdots\right.$$

Match B_ℓ to $Q/4\pi\varepsilon_0 \frac{1}{\sqrt{R^2+r^2}}$

$$B_2 = -\frac{R^2}{2} \qquad B_4 = \frac{3}{8}R^4 \qquad B_6 = -\frac{15}{48}R^6 \quad \text{etc}$$

all B_ℓ with ℓ odd vanish

2. (20) Suppose the charge density for a given atomic transition can be well approximated by $\rho(\vec{r}) = Q\frac{z}{a^4}e^{-r/a}$, where Q is the magnitude of the electron charge, and a is a size parameter $\sim 10^{-10}$m. There are positions $r \gg a$ large enough so the $\rho(\vec{r})$ can be taken to vanish. For such positions, determine the potential $V(r,\theta)$, caused by the given transition charge density. Note: $\int_0^\infty x^n\, e^{-\lambda x} = \frac{n!}{\lambda^{n+1}}$.

There is only a dipole moment term

$$V(\vec{r}) = \frac{\vec{P}\cdot\hat{r}}{4\pi\varepsilon_0 r^2} \qquad \vec{P} = \int \vec{r}\rho(\vec{r})\,d\tau$$

here $P_x = 0,\ P_y = 0$

$$\vec{P} = \hat{z}\int z\rho(\vec{r})\,d\tau = \hat{z}\int_0^\infty z^2\frac{Q}{a^4}e^{-r/a}\,4\pi r^2 dr$$

$$= \hat{z}\,\frac{4\pi}{3}\int r^4 dr\,\frac{Q}{a^4}e^{-r/a}$$

$$= \hat{z}\,\frac{4\pi}{3}\,Qa\,(24) = \hat{z}\,32\pi\,Qa$$

$$V(\vec{r}) = \frac{8}{\varepsilon_0}\frac{\cos\theta}{r^2}\,Qa$$

3. This entire lengthy problem is concerned with two concentric conducting spheres one of radius a, and the other of radius b, $(b > a)$. Furthermore, we shall only consider the region : $a \geq r \geq b$.

(a) (10) The smaller sphere carries a uniformly distributed positive charge Q,and the larger sphere carries a uniformly distributed negative charge $-Q$. There is no material in the region between the spheres. Determine the electric field in the region $a \geq r \geq b$.

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0} \frac{\hat{r}}{r^2}$$

(b) (10) For the situation of part (a) determine the potential difference between the two spheres, and make sure to state which of the spheres is at a higher potential.

$$V_a - V_b = -\int_b^a d\vec{r} \cdot \vec{E} = -\int_b^a \frac{Q}{4\pi\varepsilon_0} \frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$a < b$

$V_a - V_b > 0$ small sphere at higher potential

(c)(10) For the situation of part (a), determine the electrostatic energy of the system.

$$W = \frac{\varepsilon_0}{2}\int d^3r\, E^2 = \frac{\varepsilon_0}{2} 4\pi \int_a^b \frac{r^2 dr\, Q^2}{16\pi^2 \varepsilon_0^2 r^4}$$

$$= \frac{1}{\varepsilon_0} \frac{Q^2}{8\pi}\left[\frac{1}{a} - \frac{1}{b}\right]$$

Could also say $W = \frac{1}{2}QV$ for a capacitor with $V = V_a - V_b$

Could also say Free charge is not changed D is not changed
E is decreased by $\frac{\varepsilon_0}{\varepsilon}$ so $W = \frac{1}{2}\int D \cdot E \, d^3r$ is decreased

(d) (15) The situation of part (a) changed **only** by filling the the region between the two spheres with a uniform dielectric of permeability ϵ. Would the electrostatic energy be decreased or increased (compared with the electrostatic energy of (c))? Explain your answer.

The free charge is not changed. Thus $\vec{E}$ is decreased by a factor $\varepsilon_0/\varepsilon$. W is now $W = \frac{\varepsilon}{2}\int d^3r\, E^2$. So the net result for W

W is decreased by a factor $\frac{\varepsilon_0}{\varepsilon}$ $\left(= \frac{\varepsilon}{\varepsilon_0}\left(\frac{\varepsilon_0}{\varepsilon}\right)^2\right)$

(e) (15) Now we take a different situiation, in which the sphere at $r = 0$ is grounded, but the sphere at $r = b$ is held at a constant potential V_0. There is no dielectric material between the two spheres. Determine the potential for $a < r < b$.

$V(r) = (A_0 + B_0/r)$ since we have spherical symmetry

$V(a) = 0 = A_0 + B_0/a \Rightarrow B_0 = -aA_0$

$V(b) = V_0 = A_0 + \frac{B_0}{b} = A_0\left(1 - \frac{a}{b}\right)$

$A_0 = \frac{bV_0}{b-a}$ $\quad B_0 = -\frac{abV_0}{b-a}$

$V(r) = \frac{bV_0}{b-a}\left(1 - \frac{a}{r}\right)$

(f) (10) Consider the situation of part (e). Determine the surface charge density on the larger sphere. If you have not done (c), you may take the expression $V(r, \theta)$ as a given, and express your answer in terms of that function.

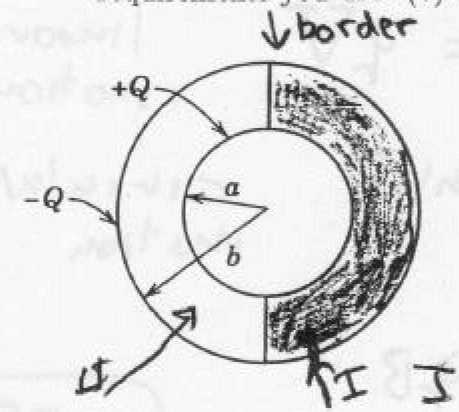
$\vec{E}\cdot\hat{n}\Big|_b = \frac{\sigma}{\varepsilon_0}$ here $\hat{n} = -\hat{r}$ $\quad \vec{E}\cdot\hat{n}\Big|_b = +\frac{\partial V}{\partial r}\Big|_b$

$\sigma = \varepsilon_0 \frac{\partial}{\partial r}\frac{V_0 b}{b-a}\left(1 - \frac{a}{r}\right)\Big|_{r=b}$

$= \frac{\varepsilon_0 V_0 b}{b-a}\frac{a}{b^2}$

V

(g) Consider the situation in part (a), but now half of the material is filled with a linear dielectric ϵ. See the figure. The goal is to compute ϕ in the region between the spheres. You can do this most easily by following the indicated steps. Consider the indicated border between the dielectric and free space. Which components of $\vec{E}$ and $\vec{D}$ are continuous at the border? (8) Show that assuming that the potential takes the form $V = \frac{K}{r}$ (where K is an unknown constant) leads to forms of $\vec{E}$ and $\vec{D}$ which are consistent with the continuity requirements you use. (8) Use Gauss law for $\vec{D}$ to determine the value of K. (4)



the tangent to the border is $\hat{r}$ and the normal is $\hat{\Theta}$

so $\vec{E}\cdot\hat{r}$ must be continuous

and $\vec{D}\cdot\hat{\Theta}$ must be continuous

If $V = K/r$

$\vec{E} = +\frac{K}{r^2}\hat{r}$ for all angles and $\vec{E}\cdot\hat{r}$ is continuous.

$\vec{D} = \frac{\epsilon_0 K \hat{r}}{r^2}$ in region II, $\vec{D} = \frac{\epsilon \hat{r}}{r^2}$ in region I

$\vec{D}\cdot\hat{\Theta} = 0$ everywhere so this is continuous

$\vec{D}\cdot\hat{n}\Big|_a = \sigma_f$ in region II $\epsilon_0 \frac{K}{a^2} = \sigma_f$ (II)

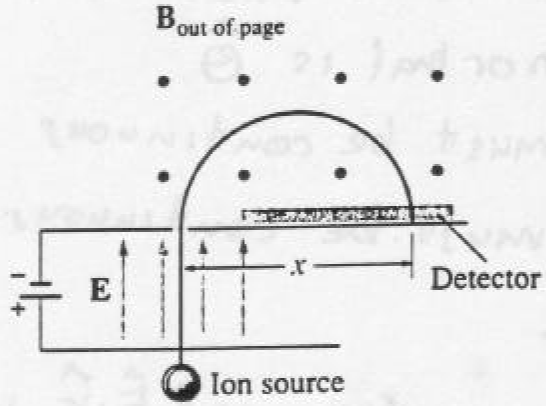
free in region I $\epsilon \frac{K}{a^2} = \sigma_f$ (I)

total charge $= Q = \int \vec{D}\cdot\hat{n}\, da$

$$= 2\pi a^2 \left(\frac{K\epsilon_0}{a^2} + \frac{K\epsilon}{a^2} \right)$$

$$K = \frac{Q}{4\pi} \frac{2}{\epsilon_0 + \epsilon}$$

4. (20) The figure shows a simple mass spectrometer, designed to analyze and separate atomic and molecular ions with different charge-to-mass ratios. In the design shown, ions accelerated (starting from rest) through a potential difference V, after which they enter a region containing a uniform magnetic field. The ions move in semicircular paths in the magnetic field, B, and land on the detector a horizontal distance x from where they entered the field region. Determine the value of x in terms of the given values of B, V and the charge q and mass m of an ion.



$\frac{1}{2}mv^2 = qV$ linear motion

$\frac{mv^2}{R} = qvB$ circular motion

$mv/R = qB$

$x = 2R$ $\quad v = \sqrt{\frac{2qV}{m}}$

$$\frac{x}{2} = \frac{qB}{mv} = \frac{qB}{m}\sqrt{\frac{m}{2qV}}$$

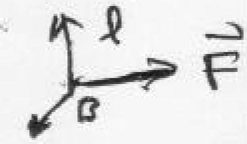
$$x = 2B\sqrt{\frac{2q}{mV}}$$

5. (a) (8) Two very long straight parallel wires separated by a distance $4D$ carry a current I in the same direction. Is the magnetic force between the two wires attractive or repulsive?

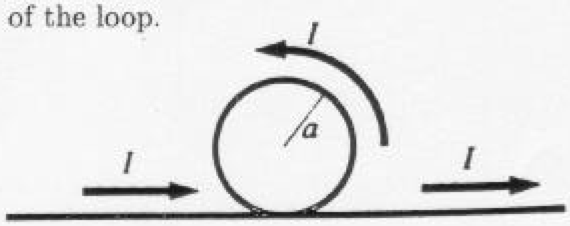
[redacted]

Attractive

wire 1 causes B out of paper at wire 2 $\vec{F} = I\,\vec{\ell} \times \vec{B}$ $\quad \vec{\ell} \uparrow$



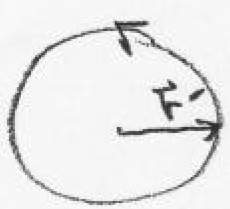
(b) (12) A single piece of wire is bent so that it includes a circular loop of radius a, see the figure. A current I flows in the direction shown. Determine the magnetic field at the center of the loop. $\vec{B}$



$B = B(\text{wire}) + B(\text{circle})$

$B_w = \dfrac{\mu_0 I}{2\pi a}$ out of paper

to get $B(\text{circle})$ use Biot Savart with $\vec{r} = 0$



$$\vec{B}_C = \frac{\mu_0}{4\pi} \int I \frac{d\vec{\ell} \times (-\vec{r}\,')}{r'^3}$$

$$= \frac{\mu_0}{4\pi} \frac{I\, 2\pi}{a} \quad \text{out of paper}$$

total $\quad B = \dfrac{\mu_0 I}{a}\left(\dfrac{1}{2\pi} + \dfrac{1}{2}\right)$