321 Midterm 2 Solutions
Prof Miller

1. (a) Consider a one-dimensional problem in which the potential, V, depends on only one variable, z. You are given $V(0) = 2$, $V(2) = -3$, and that $V(z)$ is a solution of Laplace's equation. Determine $V(z)$.

$V(z) = az + b$
$V(0) = 2 = b$
$V(2) = -3 = a2 + 2 \rightarrow a = -5/2$

$V(z) = -\frac{5}{2} z + 2$

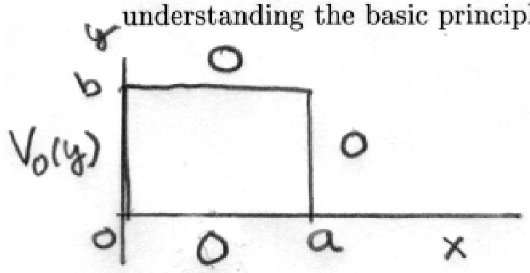
(b) One or more of the following four functions of x is a Legendre polynomial $P_l(x)$. Circle a function to indicate that it **is** a solution.

A. $\frac{1}{2}(1+x)$ B. x C. $(63x^5 - 70x^3)/8$ D. x^4

(c) Consider a region of space in which the position of a point can be defined using Cartesian (x, y, z) or spherical (r, θ, ϕ) coordinates. Determine which of the following functions is a solution of Laplace's equation ($r \neq 0$). There may be more than (or less than) one solution. Circle a function to indicate that it **is** a solution.

A. $r^4 P_4(\cos\theta)$ B z^2 C. $\frac{P_3(\cos\theta)}{r^4} + r^4 P_4(\cos\theta)$ D. $\frac{P_3(\cos\theta)}{r^3}$ E. $r^8 P_7(\cos\theta)$

2. A rectangle of sides a, b is bounded by four infinite conducting planes. The planes at $x = a, y = 0, y = b$ are grounded. The plane at $x = 0$ is held at a ~~fixed~~ potential $V_0(y)$. The aim of this problem would be to determine the potential in the region between the four planes. This would take too long, so instead we will be concerned with setting up the problem and understanding the basic principles.



(a) Determine **one** solution of Laplace's equation which is consistent with the stated boundary conditions at $x = a, y = 0, y = b$.

$$\sin\frac{n\pi y}{b}\left(e^{+\frac{n\pi}{b}(x-a)} - e^{-\frac{n\pi}{b}(x-a)}\right)$$

(b) Determine a linear superposition of solutions which could eventually be made to be consistent with all of the boundary conditions.

$$V(x,y) = \sum a_n \sin\frac{n\pi y}{b}\left(e^{\frac{n\pi}{b}(x-a)} - e^{-\frac{n\pi}{b}(x-a)}\right)$$

(c) Suppose the answer to part (b) takes the form $V(x, y) = \sum_n a_n f_n(x) g_n(y)$, where a_n are unknown coefficients, $f_n(x), g_n(y)$ are known functions and $g_n(x)$ are orthonormal on the interval $0 \le y \le b$: $\int_0^b dy\, g_n(y) g_m(y) = \delta_{nm}$. You are given $V(x = 0, y) = V_0(y)$. Determine a_n in terms of the given V_0, f_n and g_n.

$$V_0(y) = \sum a_n f_n(0)\, g_n(y)$$

$$a_n = \frac{1}{f_n(0)} \int_0^b dy\, V_0(y)\, f_n(y)$$

3. A region of charge, with unkown charge density $\rho(\vec{r})$ is localized around an origin $\vec{r} = 0$. You are given the potential $V(\vec{r})$ in the region outside the charge distribution is

$$V(r,\theta) = \frac{A}{r} + \frac{B\cos(\theta)}{r^2} + C\frac{(3\cos^2(\theta)-1)}{r^3},$$

where A, B, C are given positive constants. The aim of the problem is to deduce as much as possible about $\rho(\vec{r})$. Hint: $\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{l=0}^{\infty} \frac{(r')^l}{r^{l+1}} P_l(\hat{r} \cdot \hat{r}')$, if $r > r'$.

(a) Determine $\frac{\partial \rho(\vec{r})}{\partial \phi}$. $= 0$ (azimuthal symmetry)

(b) Determine the total charge of the system. as $r \to \infty$ $V \to \frac{Q}{4\pi\varepsilon_0 r}$

So $\boxed{Q = A\, 4\pi\varepsilon_0}$

(c) Determine the dipole moment $\vec{p}$ of the system.

next term in V as $r \to \infty$ $= \frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2}$

So $\vec{p} = B\,\hat{z}\, 4\pi\varepsilon_0$

(d) Compute the integral $\int d^3r \rho(\vec{r}) r^2 P_2(\cos\theta)$. $V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(r')}{|\vec{r}-\vec{r}\,'|}$

$V(r,\theta) = \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \int d^3r' \rho(\vec{r}\,') r'^l P_l(\hat{r}\cdot\hat{r}') \frac{1}{4\pi\varepsilon_0}$

$B_2 = C$

$V(r, \theta=0) = \sum_{l=0} \frac{1}{r^{l+1}} \int \frac{d^3r'}{4\pi\varepsilon_0} \rho(\vec{r}\,') r'^l P_l(\hat{z}\cdot\hat{r}') = \sum \frac{B_n}{r^{n+1}} P_n$

look at $n = l = 2$ $(2C\, 4\pi\varepsilon_0) = \int d^3r' \rho(\vec{r}\,') (r')^2 P_l(\cos\theta')$