

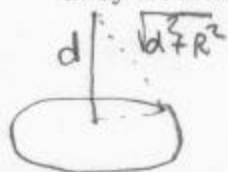
2002 - 321 final solutions
Prof Miller

1 (a) (15) Consider a ring of charge of radius R with a constant linear charge density, λ . The ring is in the xy plane, with center at the origin. Determine the total charge and dipole moment of the disk.

$$\text{charge } Q = 2\pi\lambda R$$

$$\text{dipole moment} = 0$$

(b) (10) Determine the potential at a point which is on the z -axis, a finite distance $d > R$ away from the center.



$$V(d) = \frac{2\pi\lambda R}{4\pi\epsilon_0} \frac{1}{\sqrt{d^2 + R^2}} = \frac{\lambda R}{2\epsilon_0} \frac{1}{\sqrt{d^2 + R^2}}$$

(c) (15) Now determine the potential $V(\vec{r})$, in terms of an expansion in Legendre polynomials, at any point where $r > R$. Hint: $(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \dots$. Keep the two most important non-vanishing terms.

$$V(\vec{r}) = \sum_{n=0}^{\infty} \frac{a_n P_n(\cos\theta)}{r^{n+1}} \quad \text{large } r > R$$

$$V(\theta=0, d) = \sum_{n=0}^{\infty} \frac{a_n}{d^{n+1}} = \frac{2\lambda R}{2\epsilon_0 d} \frac{1}{\sqrt{1+R^2/d^2}}$$

$$= \frac{\lambda R}{2\epsilon_0} \left(\frac{1}{d} - \frac{1}{2} \frac{R^2}{d^3} + \dots \right)$$

$$a_0 = \frac{\lambda R}{2\epsilon_0} \quad a_1 = 0 \quad a_2 = -\frac{\lambda R}{4\epsilon_0} \frac{R^2}{d^3}$$

$$V(\vec{r}) = \frac{\lambda R}{2\epsilon_0} \left(\frac{1}{r} - \frac{1}{2} \frac{R^2}{r^3} P_2(\cos\theta) \right)$$

2. (20) Suppose the charge density of a given system can be well approximated by $\rho(\vec{r}) = Q \frac{(3z^2 - r^2)}{a^5} \theta(a - r)$, where Q is quantity with dimensions of charge, a is a size parameter and $\theta(x) = 1$ for positive values of x and vanishes for other values of x . There are positions $r \gg a$ large enough so the $\rho(\vec{r})$ can be taken to vanish. For such positions, determine the non-vanishing potential $V(r, \theta)$, caused by the given charge density. Hint: what are the charge and dipole moment of the system? $\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{n=0}^{\infty} \frac{r'^n}{r^{n+1}} P_n(\hat{r} \cdot \hat{r}')$. $\int d^3r' P_n^2(\hat{r}') = \frac{2}{2n+1}$

charge = 0, (because $0 = \int d^3r \theta(a-r) (3z^2 - r^2)$)
 dipole moment = 0

Need next term.

$$V(\vec{r}) = \sum_{n \neq 1} \frac{Q_n}{r^{n+1}} P_n(\cos \theta) \quad \text{for } r \gg a$$

$$= \frac{Q_2 P_2(\cos \theta)}{r^3} \quad r \gg a$$

$$V(r, \theta=0) = \frac{Q_2}{r^3} = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} \quad (\theta=0)$$

$$= \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \int d^3r' r'^2 \rho(\vec{r}') P_2(\hat{r} \cdot \hat{r}') \quad \left. \begin{array}{l} \theta=0 \\ P_2(\hat{r} \cdot \hat{r}') \\ \text{at } \theta=0 \\ = \\ P_2(\cos \theta) \end{array} \right\}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \int_0^a r'^4 dr' \int_{-1}^1 d\cos \theta' \frac{(3\cos^2 \theta' - 1)}{2} r'^2 \int_0^{2\pi} d\phi' \frac{Q}{a^5}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{Qa^2}{7r^3} \int_{-1}^1 dx \frac{(3x^2 - 1)^2}{2} = \frac{1}{4\pi\epsilon_0} \frac{Qa^2}{7r^3} \frac{2}{2} \left(\frac{1}{5} - \frac{6}{3} + 1 \right)$$

$$= \frac{Qa^2}{4\pi\epsilon_0} \frac{1}{7} \frac{1}{r^3} \frac{4}{5}$$

$$V(\vec{r}) = \frac{Qa^2}{4\pi\epsilon_0} \frac{4}{35} \frac{P_2(\cos \theta)}{r^3}$$

3. A very long wire is placed along the z axis, with current I flowing upward. A point P is a distance s away from the wire.

(a) (12) Obtain an expression for the vector potential \vec{A} at the point P . Your answer should be in the form of an integral in which **every** symbol is defined. Do not evaluate the integral.



$$\vec{A}(P) = \frac{\mu_0}{4\pi} I \hat{z} \int_{-\infty}^{\infty} \frac{dz'}{\sqrt{z'^2 + s^2}}$$

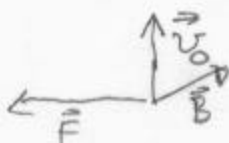
(b) (8) The magnetic field has at the point P has a magnitude $B_0 = \frac{\mu_0 I}{2\pi s}$. State the direction of the magnetic field.

\vec{B} points into the page

(c) (5) A particle of mass m and charge $Q > 0$ is placed at rest at the point P . Determine what happens to the charged particle, and state your answer in **one** circled sentence.

The particle stays where it is.

(d) (5) A particle of mass m and charge $Q > 0$ is placed at the point P , moving upward with an initial speed v_0 . Determine what happens to the charged particle. The quantity v_0 is very small: $v_0 \ll \frac{\mu_0 Q I}{2\pi m}$.



There is an attractive force towards the wire. This causes circular motion ^(in the plane \perp to wire) in a radius R

given by $q v_0 B = \frac{m v_0^2}{R}$

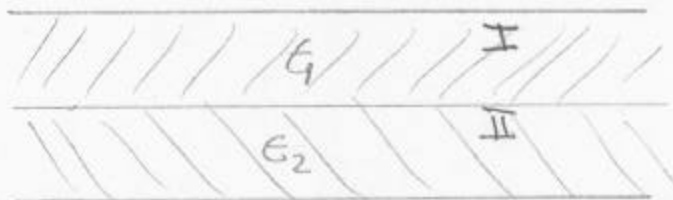
$R = \frac{m v_0}{q B}$ with v_0 small as given $R \ll s$

So B can be taken as constant

You can also get this result by solving $\vec{F} = q(\vec{v} \times \vec{B})$ in cylindrical coordinates

4. Two isolated square conducting plates, of side L and separation d are charged with surface densities $+\sigma > 0$ on the upper plate and $-\sigma$ on the lower plate. Two dielectric slabs, each of thickness $d/2$ and area $L \times L$ are inserted between the plates, one slab above the other as in the figure. The permeabilities are ϵ_1 and ϵ_2 . Assume that $d \ll L$. Determine:

- (12) \vec{D} everywhere between the plates.
- (11) \vec{E} everywhere between the plates.
- (15) The bound surface charges on the three dielectric surfaces.
- (11) The capacitance.
- (11) The electrostatic energy stored.



(a) $\vec{D} = \sigma \hat{z}$ pointing downwards

(b) $\vec{E}(\text{I}) = \vec{D}/\epsilon_1$, $\vec{E}(\text{II}) = \vec{D}/\epsilon_2$

(c) $\vec{P} = \epsilon_0 \left(\frac{\epsilon}{\epsilon_0} - 1 \right) \vec{E}$. There are 3 surfaces \rightarrow top, middle, bottom.

at T $\sigma_b = \hat{n}_T \cdot \vec{P}(\text{I}) = - \left(\frac{\epsilon_1 - \epsilon_0}{\epsilon_1} \right) \frac{\sigma}{\epsilon_1}$ \hat{n}_T is up, $\vec{P}(\text{I, II})$ are down

at B $\sigma_b = \hat{n}_B \cdot \vec{P}(\text{II}) = + \left(\frac{\epsilon_2 - \epsilon_0}{\epsilon_2} \right) \frac{\sigma}{\epsilon_2}$ \hat{n}_B is down

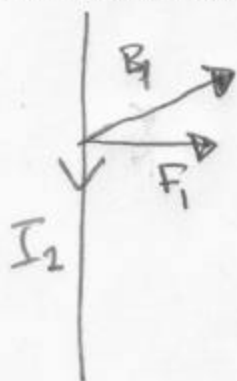
at the middle $\sigma_b = \frac{\epsilon_1 - \epsilon_0}{\epsilon_1} \sigma - \left(\frac{\epsilon_2 - \epsilon_0}{\epsilon_2} \right) \sigma$

since total bound charge must vanish

(d) $C = \frac{Q}{V} = \frac{\sigma L^2}{\frac{d}{2} (\frac{\sigma}{\epsilon_1} + \frac{\sigma}{\epsilon_2})} = \frac{2\sigma L^2}{d \left(\frac{\sigma}{\epsilon_1} + \frac{\sigma}{\epsilon_2} \right)} = \frac{2L^2}{d} \frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2}$

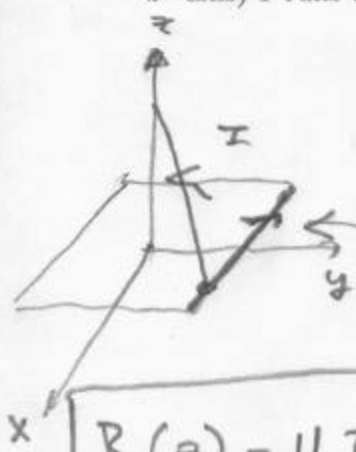
(e) $U = \frac{1}{2} C V^2 = \frac{1}{2} \frac{2L^2}{d} \frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2} \frac{d^2 \sigma^2}{4} \left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} \right)^2 = \frac{dL^2}{4} \sigma^2 \left(\frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2} \right)^{-1}$

5. (a) (8) Using pictures showing the directions of the currents, fields, and forces, prove that parallel but opposite currents repel each other.



B_1 of I_1 at I_2 points into the page
 The right hand rule gives a rightward directed horizontal force, which is repulsive.
 B_2 of I_2 at I_1 points into page and the rh rule give a leftward directed force, which is repulsive

- (b) (12) A square loop with sides of length $2a$ lies in the xy plane with its center at the origin and sides parallel to the x and y axes. A counterclockwise current (if one looks down the z -axis) I runs around the loop. Determine the magnetic field $\vec{B}(z)$ on the z axis. Leave as integral



\vec{B} only has a z -component, since the horizontal components caused by the four straight segments cancel. Furthermore each segment gives the same contribution. So we need only consider the segment with the heavy line

$$B_z = \frac{4\mu_0 I}{4\pi} \int_{-a}^a dy' \frac{(-\hat{i}) \times (z\hat{k} - x'\hat{i} - a\hat{j}) \cdot \hat{k}}{[z^2 + a^2 + y'^2]^{3/2}}$$

$$B_z(z) = \frac{\mu_0 I}{\pi} \int_{-a}^a \frac{dy' a}{[z^2 + a^2 + y'^2]^{3/2}}$$

- (c) (10) Consider a magnetic field $\vec{B} = axy\hat{i} + by^2\hat{j}$. Determine the relation between the constants a and b .

$$\vec{\nabla} \cdot \vec{B} = 0 = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = ay + 2by = 0$$

$$a + 2b = 0$$