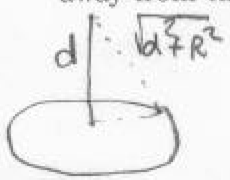
2002 - 321 final solutions
Prof Miller

1 (a) (15) Consider a **ring** of charge of radius R with a constant linear charge density, λ. The ring is in the xy plane, with center at the origin. Determine the total charge and dipole moment of the disk.

Charge $Q = 2\pi\lambda R$

dipole moment $= 0$

(b) (10) Determine the potential at a point which is on the $z-$axis, a finite distance $d > R$ away from the center.



$$V(d) = \frac{2\pi\lambda R}{4\pi\varepsilon_0}\frac{1}{\sqrt{d^2+R^2}} = \frac{\lambda R}{2\varepsilon_0}\frac{1}{\sqrt{d^2+R^2}}$$

(c)(15) Now determine the potential $V(\vec{r}\,)$, in terms of an expansion in Legendre polynomials, at any point where $r > R$. Hint: $(1+x)^{-1/2} = 1 - \frac{1}{2}\,x + \frac{3}{8}\,x^2 - \frac{5}{16}\,x^3 + \cdots$. Keep the two most important non-vanishing terms.

$$V(\vec{r}) = \sum_{n=0}^{\infty} \frac{a_n P_n(\cos\theta)}{r^{n+1}} \qquad \text{large } r > R.$$

$$V(\theta=0, d) = \sum_{n=0}^{\infty} \frac{a_n}{d^{n+1}} = \frac{\lambda R}{2\varepsilon_0 d}\frac{1}{\sqrt{1+R^2/d^2}}$$

$$= \frac{\lambda R}{2\varepsilon_0}\left(\frac{1}{d} - \frac{1}{2}\frac{R^2}{d^3} + \cdots\right)$$

$$a_0 = \frac{\lambda R}{2\varepsilon_0} \qquad a_1 = 0 \qquad a_2 = -\frac{\lambda R}{4\varepsilon_0}\frac{R^2}{d^3}$$

$$V(\vec{r}) = \frac{\lambda R}{2\varepsilon_0}\left(\frac{1}{r} - \frac{1}{2}\frac{R^2}{r^3}P_2(\cos\theta)\right)$$

2. (20) Suppose the charge density of a given system can be well approximated by $\rho(\vec{r}) = Q\frac{(3z^2-r^2)}{a^5}\theta(a-r)$, where Q is quantity with dimensions of charge, a is a size parameter and $\theta(x) = 1$ for positive values of x and vanishes for other values of x. There are positions $r \gg a$ large enough so the $\rho(\vec{r})$ can be taken to vanish. For such positions, determine the non-vanishing potential $V(r,\theta)$, caused by the given charge density. Hint: what are the charge and dipole moment of the system? $\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{n=0}^{\infty} \frac{r'^n}{r^{n+1}} P_n(\hat{r}\cdot\hat{r}\,')$. $\int_{-1}^{1} dx\, P_n^2(x) = \frac{2}{2n+1}$

charge = 0, (because $0 = \int d^3r\, \theta(a-r)\,(3z^2-r^2)$)

dipole moment = 0

need next term.

$$V(\vec{r}) = \sum \frac{a_n}{r^{n+1}} P_n(\cos\theta) \qquad \text{for } r > a$$

$$= \frac{a_2 P_2(\cos\theta)}{r^3} \qquad r \gg a$$

$$V(r,\theta=0) = \frac{a_2}{r^3} = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\Bigg|_{\theta=0}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\int d^3r'\, r'^2 \rho(\vec{r}\,')\, P_2(\hat{r}'\cdot\hat{r})\Bigg|_{\theta=0}$$

[$P_2(\hat{r}\cdot\hat{r}')$ at $\theta=0$ $= P_2(\cos\theta')$]

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\int_0^a r'^4 dr' \int_{-1}^{1} d\cos\theta' \left(\frac{3\cos^2\theta'-1}{2}\right)^2 2\int_0^{2\pi} d\phi' \frac{Q}{a^5}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{Qa^2}{7r^3}\int_{-1}^{1} dx \frac{(3x^2-1)^2}{2} = \frac{1}{4\pi\varepsilon_0}\frac{Qa^2}{7r^3}\frac{2}{2}\left(\frac{9}{5}-\frac{6}{3}+1\right)$$

$$= \frac{Qa^2}{4\pi\varepsilon_0}\frac{1}{7}\frac{1}{r^3}\frac{4}{5}$$

$$V(\vec{r}) = \frac{Qa^2}{4\pi\varepsilon_0}\frac{4}{35}\frac{P_2(\cos\theta)}{r^3}$$

3. A very long wire is placed along the z axis, with current I flowing upward. A point P is a distance s away from the wire.

(a) (12) Obtain an expression for the vector potential $\vec{A}$ at the point P. Your answer should be in the form of an integral in which **every** symbol is defined. Do not evaluate the integral.

$$\vec{A}(P) = \frac{\mu_0}{4\pi} I \hat{z} \int_{-\infty}^{\infty} \frac{dz'}{\sqrt{z'^2 + s^2}}$$

(b) (8) The magnetic field has at the point P has a magnitude $B_0 = \frac{\mu_0 I}{2\pi s}$. State the direction of the magnetic field.

$\vec{B}$ points into the page

15

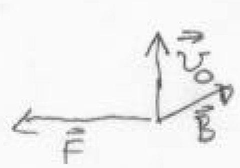
(c) ([redacted]) A particle of mass m and charge $Q > 0$ is placed at rest at the point P. Determine what happens to the charged particle, and state your answer in **one** circled sentence.

The particle stays where it is.

15

(d) ([redacted]) A particle of mass m and charge $Q > 0$ is placed at the point P, moving upward with an initial speed v_0. Determine what happens to the charged particle. The quantity v_0 is very small: $v_0 \ll \frac{\mu_0 QI}{2\pi m}$.

There is an attractive force towards the wire. This causes circular motion (in the plane ⊥ to wire) in a radius R given by $qv_0 B = \frac{mv_0^2}{R}$



$R = \frac{mv_0}{qB}$ with V_0 small as given $R \ll s$ so B can be taken as constant.

You can also get this result by solving $\vec{F} = q(\vec{v} \times \vec{B})$ in cylindrical coordinates

4. Two isolated square conducting plates, of side L and separation d are charged with surface densities $+\sigma > 0$ on the upper plate and $-\sigma$ on the lower plate. Two dielectric slabs, each of thickness $d/2$ and area $L \times L$ are inserted between the plates, one slab above the other as in the figure. The permeabilities are ϵ_1 and ϵ_2. Assume that $d \ll L$. Determine:

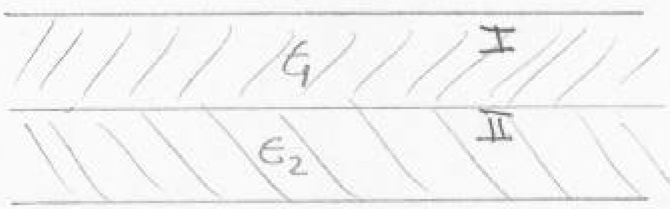
(a) (12) **D** everywhere between the plates.

(b) (11) **E** everywhere between the plates.

(c) (15) The bound surface charges on the three dielectric surfaces.

(d) (11) The capacitance.

(e) (11) The electrostatic energy stored.



(a) $\vec{D} = \sigma$ pointing downwards

(b) $\vec{E}(I) = \vec{D}/\epsilon_1$, $\vec{E}(II) = \vec{D}/\epsilon_2$

(c) $\vec{P} = \epsilon_0\left(\frac{\epsilon}{\epsilon_0} - 1\right)\vec{E}$. There are 3 surfaces top T, middle M, bottom B.

$\hat{n}_T$ is up , $\vec{P}(I, II)$ are down

at T $\quad \sigma_b = \hat{n}_T \cdot \vec{P}(I) = -(\epsilon_1 - \epsilon_0)\frac{\sigma}{\epsilon_1}$

at B $\quad \sigma_b = \hat{n}_B \cdot P(II) = +(\epsilon_2 - \epsilon_0)\frac{\sigma}{\epsilon_2}$ $\quad \hat{n}_B$ is down

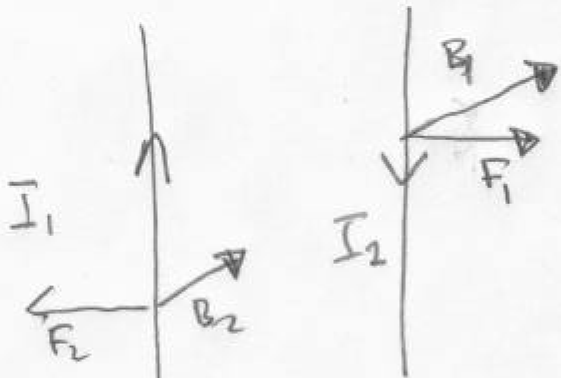
at the middle $\quad \sigma_b = \frac{\epsilon_1 - \epsilon_0}{\epsilon_1}\sigma - \left(\frac{\epsilon_2 - \epsilon_0}{\epsilon_2}\right)\sigma$

since total bound charge must vanish

(d) $C = \frac{Q}{V} = \frac{\sigma L^2}{\frac{d}{2}(E_1 + E_2)} = \frac{2\sigma L^2}{d\left(\frac{\sigma}{\epsilon_1} + \frac{\sigma}{\epsilon_2}\right)} = \frac{2L^2}{d}\frac{\epsilon_1\epsilon_2}{\epsilon_1 + \epsilon_2}$

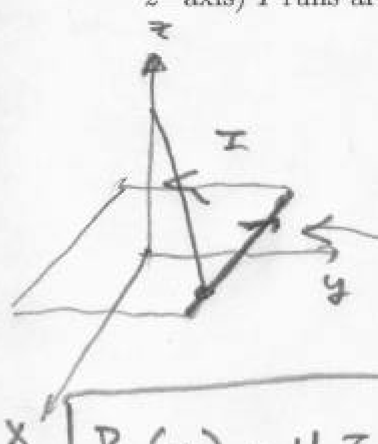
(e) $U = \frac{1}{2}CV^2 = \frac{1}{2}\frac{2L^2}{d}\frac{\epsilon_1\epsilon_2}{\epsilon_1+\epsilon_2}\frac{d^2}{4}\sigma^2\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}\right)^2 = \frac{dL^2}{4}\sigma^2\left(\frac{\epsilon_1\epsilon_2}{\epsilon_1+\epsilon_2}\right)^{-1}$

5. (a) (8) Using pictures showing the directions of the currents, fields, and forces, prove that parallel but opposite currents repel each other.



B_1 of I_1 at I_2 points into the page
The right hand rule gives a rightward directed horizontal force, which is repulsive
B_2 of I_2 at I_1 points into page and the rh rule give a leftward directed force, which is repulsive

(b) (12) A square loop with sides of length $2a$ lies in the xy plane with its center at the origin and sides parallel to the x and y axes. A counterclockwise current (if one looks down the $z-$axis) I runs around the loop. Determine the magnetic field $\vec{B}(z)$ on the z axis. Leave as integral



$\vec{B}$ only has a z-component, since the horizontal components caused by the four straight segments cancels. Furthermore each segment gives the same contribution. So we need only consider the segment with the heavy line

$$B_z = 4\frac{\mu_0 I}{4\pi}\int_{-a}^{a} dy' \frac{(-\hat{\imath})\times(z\hat{k} - x'\hat{\imath} - a\hat{\jmath})\cdot\hat{k}}{[z^2+a^2+y'^2]^{3/2}}$$

$$\boxed{B_z(z) = \frac{\mu_0 I}{\pi}\int_{-a}^{a}\frac{dy'\,a}{[z^2+a^2+y'^2]^{3/2}}}$$

(c) (10) Consider a magnetic field $\mathbf{B} = axy\hat{\mathbf{i}} + by^2\hat{\mathbf{j}}$. Determine the relation between the constants a and b.

$$\vec{\nabla}\cdot\vec{B} = 0 = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = ay + 2by = 0$$

$$\boxed{a + 2b = 0}$$