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Electromagnetism, Physics 321 **First midterm** **8.30 am, October 24, 2003**
Autumn 2003 Instructor: David Cobden

You have 50 minutes. Begin and end on the buzzer. Answer all questions. Please do not leave within the last ten minutes of the exam.

Write your name on every page and your ID on the first page.

Write all your working on these question sheets. Use this cover page for extra working (you might get credit for it.)

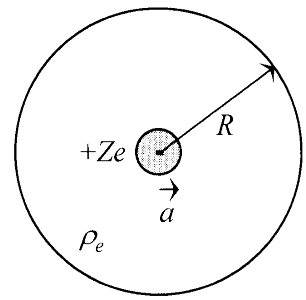
Watch the blackboard for corrections or clarifications during the exam.

This is a closed book exam. **No notes allowed. NO CALCULATORS!**

Standard notation for spherical coordinates is used throughout. Thus, for example, r is always the distance from the origin. As usual, take $V = 0$ at $r =$.

Make sure you attempt all 14 questions!

Do not turn this page until the buzzer goes at 8.30.



Consider the following model of an atom. The nucleus is a uniformly charged sphere of radius a and total charge $+Ze$. The electron cloud around it forms another sphere, with uniform charge density ρ_e and radius $R >> a$. Initially, the atom is neutral and the nucleus is at its center.

1. [5] Find ρ_e. Neglect the volume of the nucleus.

Total electronic charge $= -Ze$

$$\therefore \frac{4}{3}\pi R^3 \rho_e = -Ze \qquad \therefore \rho_e = \frac{-3Ze}{4\pi R^3}$$

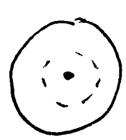
2. [5] Use Gauss's law to find the electric field vector $\mathbf{E}(\mathbf{x})$ outside the atom, ie, for $r > R$.

radial symmetry $\rightarrow \vec{E} = E(r)\hat{r}$

Gauss : $4\pi r^2 E(r) = \frac{Q}{\varepsilon_0} = 0$

$\therefore \vec{E} = 0$ for $r > R$

3. [10] Use Gauss's law to find $\mathbf{E}(\mathbf{x})$ within the electron cloud, ie, in the region $R > r > a$.



Gauss again: $4\pi r^2 E(r) = \frac{Q}{\varepsilon_0} = \left(+Ze - \frac{4}{3}\pi r^3 \rho_e\right)/\varepsilon_0$

$$= +\frac{Ze}{\varepsilon_0}\left(1 - r^3/R^3\right)$$

$$\therefore \vec{E}(\vec{x}) = \frac{Ze}{4\pi r^2 \varepsilon_0}\left(1 - \frac{r^3}{R^3}\right)\hat{r} = \frac{Ze}{4\pi\varepsilon_0}\left(\frac{1}{r^2} - \frac{r}{R^3}\right)\hat{r}$$

4. [10] Find the electric potential $V(r)$ at the surface of the nucleus, ie, at $r = a$.

$$V = \int_\infty^a -E(r)\,dr = \int_R^a -\frac{Ze}{4\pi\varepsilon_0}\left(\frac{1}{r^2} - r/R^3\right) dr$$

$$= \frac{Ze}{4\pi\varepsilon_0}\left[+\frac{1}{r} + \frac{r^2}{2R^3}\right]_R^a$$

$$= \frac{Ze}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{R} + \frac{a^2}{2R^3} - \frac{R^2}{2R^3}\right) = \frac{Ze}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{3}{2R} + \frac{a^2}{2R^3}\right)$$

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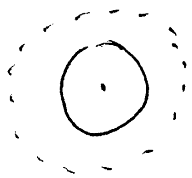
The atom is now converted into a positive ion by removing one electron.

5. [5] What now is the electric field **E**(**x**) outside the ion, ie, for $r > R$?



Gauss: $4\pi r^2 E(r) = \dfrac{+Ze - (Z-1)e}{\varepsilon_0} = \dfrac{+e}{\varepsilon_0}$

$\therefore \vec{E} = \dfrac{+e}{4\pi\varepsilon_0 r^2}\hat{r}$ for $r > a$

6. [5] State the boundary conditions on **E** at the surface of the ion. $(\vec{E}_1, \vec{E}_2)$

$E_{\parallel}$ just inside $= E_{\parallel}$ just outside $\left(\begin{array}{l} E_{\phi 1} = E_{\phi 2} \\ E_{\theta 1} = E_{\theta 2} \end{array}\right)$

$E_{\perp}$ just inside $= E_{\perp}$ just outside (because $\sigma = 0$)

so $E_{r1} = E_{r2}$

7. [10] Calculate the energy stored in the electric field outside the ion.

$$U = \int_{r>R} \frac{1}{2}\varepsilon_0 E(\vec{x})^2 \, d^3x$$

$$= \int_{r=R}^{\infty} \frac{1}{2}\varepsilon_0 \left(\frac{e}{4\pi\varepsilon_0 r^2}\right)^2 4\pi r^2 dr$$

$$= 4\pi \cdot \frac{\varepsilon_0}{2} \cdot \left(\frac{e}{4\pi\varepsilon_0}\right)^2 \int_R^{\infty} \left(\frac{1}{r^2}\right) r^2 dr$$

$$= \frac{e^2}{8\pi\varepsilon_0}\left[-\frac{1}{r}\right]_R^{\infty} = \frac{e^2}{8\pi\varepsilon_0 R}$$

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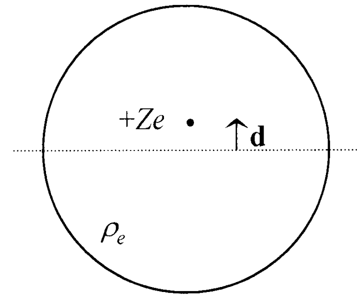
Let's return to the neutral atom, and consider the limit $a \to 0$, so the nuclear charge density can be represented by a δ-function.



The atom is now polarized, by displacing the nucleus away from the center of the electron cloud by a vector **d**, where $d << R$.

8. [5] Write down expressions, involving a 3D δ-function, for the total charge density (electrons plus nucleus) for $r \leq R$ and $r > R$.

$$\rho = +Ze\,\delta^3(\vec{x}-\vec{d}) + \rho_e \quad \text{for } r \leq R$$

$$= 0 \quad r > R$$

$$\rho_e = \frac{-3Ze}{4\pi R^3}$$

9. [6] Carefully write down the general integral relating the potential $V(\mathbf{x})$ at any point in space to the total charge density $\rho(\mathbf{x})$. Identify precisely the Green's function in the expression.

$$V(\vec{x}) = \int_{\text{all space}} \underbrace{\frac{1}{4\pi|\vec{x}-\vec{x}'|}}_{\text{Green's function}} \frac{\rho(\vec{x}')}{\varepsilon_0}\, d^3x'$$

10. [15] Evaluate this integral for the charge density in Q. 8, to show that the potential for $r > R$ is equivalent to that of a dipole. [~~This should only take three line~~s. Each part of the integral can be done in a single step, using the properties of the δ-function and Gauss's law. ~~Give a few words of explanation in each case~~.]

$$V(\vec{x}) = \int_{r \leq R} \frac{Ze\,\delta^3(\vec{x}-\vec{d}) + \rho_e}{4\pi\varepsilon_0|\vec{x}-\vec{x}'|}\, d^3x'$$

using properties of δ-function

$$= \frac{Ze}{4\pi\varepsilon_0|\vec{x}-\vec{d}|} + \int_{\text{sphere}} \frac{\rho_e}{4\pi\varepsilon_0|\vec{x}-\vec{x}'|}\, d^3x'$$

this is just the potential of the uniformly charge electron sphere

$$= \frac{+Ze}{4\pi\varepsilon_0|\vec{x}-\vec{d}|} + \frac{-Ze}{4\pi\varepsilon_0|\vec{x}|} = \frac{Ze}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{x}-\vec{d}|} - \frac{1}{|\vec{x}|}\right]$$

This is the potential of a dipole $\vec{d}\uparrow$ +Ze at $\vec{d}$, −Ze at O

11. [4] What is the dipole moment **p** of the atom, and where is it centered?

$$\vec{p} = Ze\vec{d} \qquad \text{center is at } \frac{\vec{d}}{2}.$$

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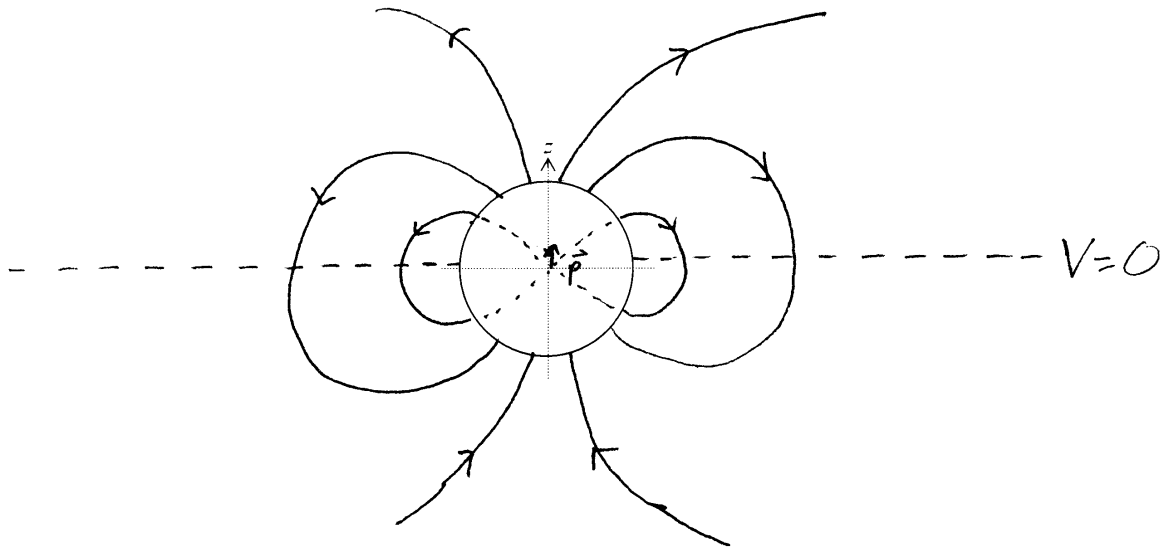
12. [8] Take **p** to be along the z-axis. Sketch below the electric field lines outside the polarized atom. Also sketch the equipotential $V(\mathbf{x}) = 0$ outside the atom as a dashed line on the same diagram.



13. [12] A proton (charge $+e$) lies on the z-axis at a distance z from the center of the atom. Show that the force on the proton is given approximately by $\mathbf{F}(z) = \dfrac{Ze^2 d\hat{\mathbf{k}}}{2\pi\varepsilon_0 z^3}$.

[Use the fact that the potential of a dipole at the origin is $V(\mathbf{x}) = \mathbf{p}.\hat{\mathbf{r}}/4\pi\varepsilon_0 r^2$. This is a quick calculation if you employ symmetry and simplify V at the start.]

$\vec{F} = +e\vec{E}$ $\qquad\qquad$ $\vec{p} = p\hat{k} = Zed\hat{k}$

By symmetry, $\vec{E} = E(z)\hat{k} = -\dfrac{\partial V}{\partial z}\hat{k}$

$\therefore \vec{F} = e\left(\dfrac{-\partial V}{\partial z}\right)\hat{k} = -e\hat{k}\dfrac{\partial}{\partial z}\left(\dfrac{\vec{p}.\hat{r}}{4\pi\varepsilon_0 r^2}\right)$ at $\vec{x} = (0, 0, z)$

$r = z$

$\vec{p}.\hat{r} = p$

$= -e\hat{k}\dfrac{\partial}{\partial z}\left(\dfrac{Zed}{4\pi\varepsilon_0 z^2}\right)$

$= \dfrac{Ze^2 d}{2\pi\varepsilon_0 z^3}\hat{k}$ $\qquad$ big z and small z