 Name ________________________________

**Electromagnetism, Physics 321** **Second midterm** **8.20 am, November 21, 2003**
**Autumn 2003** Instructor: David Cobden

You have 60 minutes. End on the buzzer. Answer all questions. Please do not leave within the last ten minutes of the exam.

Write your name on every page and your ID on the first page.

Write all your working on these question sheets. Use this cover page for extra working (you might get credit for it.)

Watch the blackboard for corrections or clarifications during the exam.

This is a closed book exam. **No notes allowed. NO CALCULATORS!**

Standard notation for spherical coordinates is used throughout. Thus, for example, $r$ is always the distance from the origin. As usual, take $V = 0$ at $r =$ .

Make sure you attempt all 14 questions!

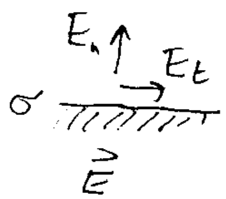
**Do not turn this page until I say 'go'.**

**Useful metric conversions**

1 trillion microphones = 1 megaphone
1 millionth of a fish = 1 microfiche
1 trillion pins = 1 terrapin
10 rations = 1 decoration
10 millipedes = 1 centipede
3 1/3 tridents = 1 decadent
2 monograms = 1 diagram

 Name solutions

1. [10] Explain why the electric field at the surface of a conductor must be normal to the surface, starting with the assumption that a conductor contains charges which are free to move, and ending with the relationship between the field and the surface charge density.




$\vec{E} = 0$ inside conductor, or charge would rearrange to make it so.

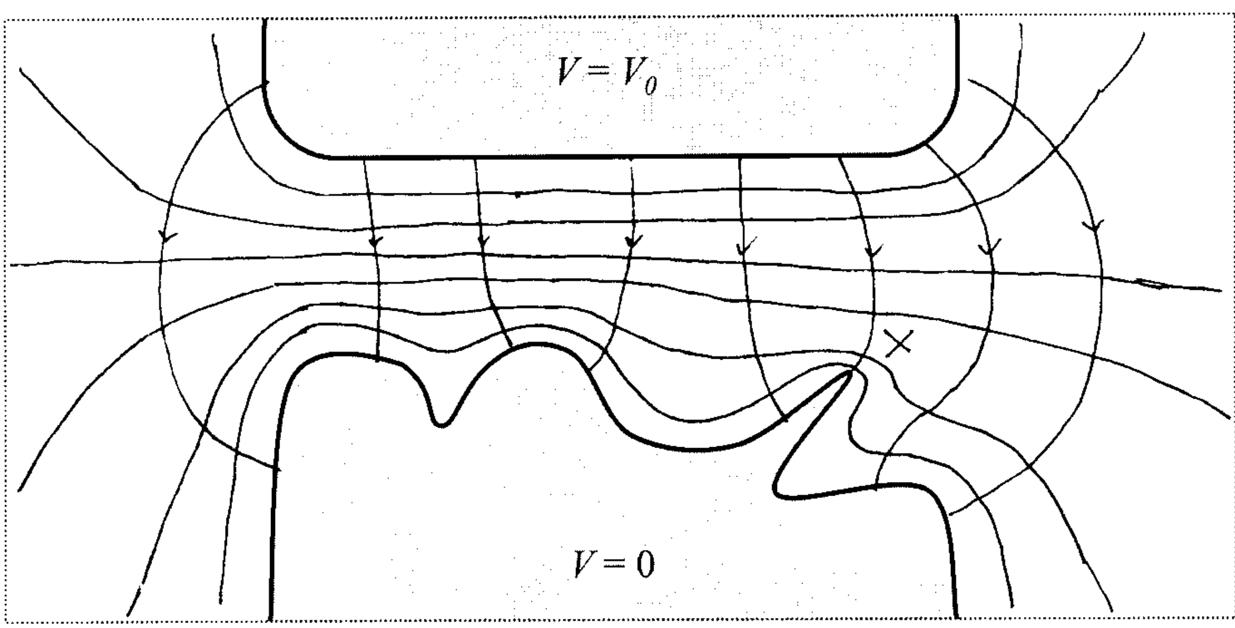
tangential b.c. on $\vec{E}$ (from $\vec{\nabla} \times \vec{E} = 0$) $\rightarrow E_t = 0$

normal b.c. on $\vec{E}$ (from $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$) $\rightarrow E_n = \frac{\sigma}{\varepsilon_0}$

2. [5] State the uniqueness theorem for Laplace's equation in a region $\mathcal{V}$ bounded by surfaces on which either the potential $V$ or its normal derivative $\partial V/\partial n$ is specified. $\nabla^2 V = 0$

If you have any solution $V(\vec{x})$ of in $\mathcal{V}$ which matches the b.c.'s then it must be the correct solution.

3. [9] The adjacent sides of two conductors are shown here in cross-section. They are uniform in the direction perpendicular to the paper. A potential $V_0$ is applied to the upper while the lower is grounded. Sketch equipotentials and a few field lines within the region indicated by the dotted box.

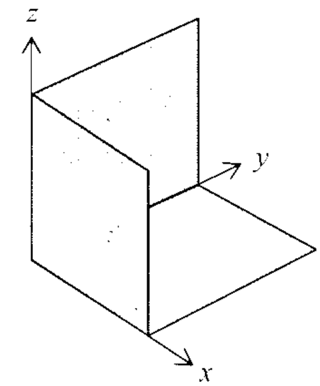



4. [5] As $V_0$ is increased, electrical breakdown occurs at the point where the electric field is maximum at the surface. Mark this point with an $X$ and explain how you know the field is strongest there.

Equipotentials are closest there.
(Field is higher at strongly curved surface)

 Name solutions

Three perpendicular, infinite, conducting planes at potential $V_0$ form a 'cubic corner'. Let them be the planes $x = 0$, $y = 0$ and $z = 0$, and let $\mathcal{V}$ be the region $x > 0, y > 0, z > 0$.




5. [8] Show that $V(\mathbf{x}) = (ax+b)(cy+d)(ez+f)$ obeys Laplace's equation in $\mathcal{V}$.

$$\nabla^2 V = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(ax+b)(cy+d)(ez+f)$$

$$= 0 + 0 + 0$$

$$= 0$$

6. [8] Hence show that the potential in $\mathcal{V}$ is given by $V(\mathbf{x}) = V_0 + Cxyz$, where $C$ is a constant.

At $x = 0$, $V(\vec{x}) = V_0$
and at $y = 0$
and at $z = 0$ $\therefore V(\vec{x})$ obeys the b.c.'s

$\therefore$ by uniqueness, it must be the correct solution.

7. [5] Find the surface charge density at position $(x,y,0)$ on the $z$=0 surface.

$$\sigma = \varepsilon_0 E_n = -\varepsilon_0 \frac{\partial V}{\partial z} = -\varepsilon_0 C xy$$

8. [5] Find the electrostatic pressure exerted on the surface at the same position. What is its direction?

Upwards (towards +ve $z$)

$$P = \frac{1}{2}\varepsilon_0 E_n^2 = \frac{1}{2\varepsilon_0}\sigma^2 = \frac{\varepsilon_0}{2} C^2 x^2 y^2$$

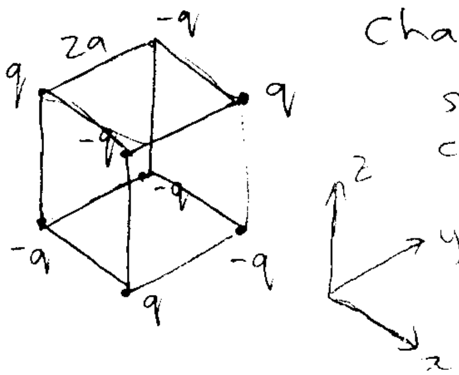
9. [5] Explain why $V$ diverges at infinity and why this unphysical behavior is not found in any real system.

Total charge on plates is infinite.
In reality, everything is finite in size and in charge and $V \to 0$ for large $r$. Our solution only works on scales much smaller than the size of the plates.

 Name solutions

The planes are now *grounded*, and a point charge $+q$ is placed at position $\boldsymbol{x} = (a,a,a)$.

10. [10] Describe a set of 7 image charges which may be used to find the potential $V(\boldsymbol{x})$.




Charges are at corners of a cube of side 2a. centered at origin.

$+q$ at $(a,a,a)$ is real charge
$-q$ " $(a,a,-a)$
$-q$ at $(-a,a,a)$
$-q$ at $(a,-a,a)$
$+q$ at $(-a,-a,a)$
$+q$ at $(-a,a,-a)$
$+q$ at $(a,-a,-a)$
$-q$ at $(-a,-a,-a)$

11. [5] What is the total induced surface charge on the $x = 0$ plane? (No integration is necessary!)

Total induced surface charge $= \Sigma$(image charges)
$= -q$ (by Gauss)

by symmetry, it must be $\frac{-q}{3}$ on each plane.

12. [6] If $V$ is expanded as a power series in $r$ for $r > \sqrt{3}a$, which term, ie, which power of $r$, will dominate at large $r$? Why? (Do not try to calculate the term.)

It's two equal & opposite quadrupoles, making an octopole.

$$V = \frac{A(\theta,\phi)}{r} + \frac{B(\theta,\phi)}{r^2} + \frac{C(\theta,\phi)}{r^3} + \frac{D(\theta,\phi)}{r^4} + \ldots$$

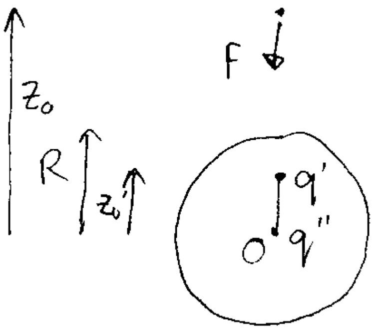
(first three terms crossed out, each marked 0)

monopole: 0, dipole: 0, quadrupole: 0, octopole

$\therefore$ At large $r$, $V = \frac{D(\theta,\phi)}{r^4}$

Page 5 Name solutions

13. [14] An electron of charge $e$ is located at a distance $z_0$ from the center of a *neutral, isolated,* conducting sphere of radius $R$. Find the force of attraction of the electron towards the sphere.




To make sphere an equipotential we need image charge

$$q' = -\frac{R}{z_0}e \quad \text{at } z_0' = \frac{R^2}{z_0} \text{ from sphere center (origin)}$$

To make sphere neutral we need

$$q'' = -q' = \frac{R}{z_0}e \quad \text{at origin also.}$$

(Gauss: total $Q_{enc}$ in sphere surrounding conductor $= q' + q'' = 0$)

$$F = -eE = -e\left[\frac{q''}{4\pi\varepsilon_0 z_0^2} + \frac{q'}{4\pi\varepsilon_0 (z_0 - z_0')^2}\right] = \frac{-e}{4\pi\varepsilon_0}\left[\frac{eR/z_0}{z_0^2} - \frac{eR/z_0}{(z_0 - z_0')^2}\right]$$

$$= \frac{e}{4\pi\varepsilon_0}\cdot\frac{Re}{z_0}\left[\frac{1}{(z_0 - z_0')^2} - \frac{1}{z_0^2}\right] = \frac{Re^2}{4\pi\varepsilon_0 z_0^3}\left[\frac{1}{\left(1 - \frac{R^2}{z_0^2}\right)^2} - 1\right]$$

check: at $z_0 = R$ it diverges
at $R = 0$ $F \to 0$

14. [5] The electron finally hits the sphere and merges into it. What is the final potential of the sphere?

$$V = \frac{q}{C} = \frac{e}{4\pi\varepsilon_0 R}$$

( e includes the minus sign!)



THE END