Name ____________solutions________________

Electromagnetism, Physics 321 **Final exam** **8.30 am, December 15, 2003**
Autumn 2003 Instructor: David Cobden

Do not turn this page until the buzzer goes at 8.30. You have 110 minutes. End on the buzzer at 10.20.

This exam has 20 questions, devided into three sections. Attempt all of them. Each section is worth 66 points. You have about 35 minutes for each section.

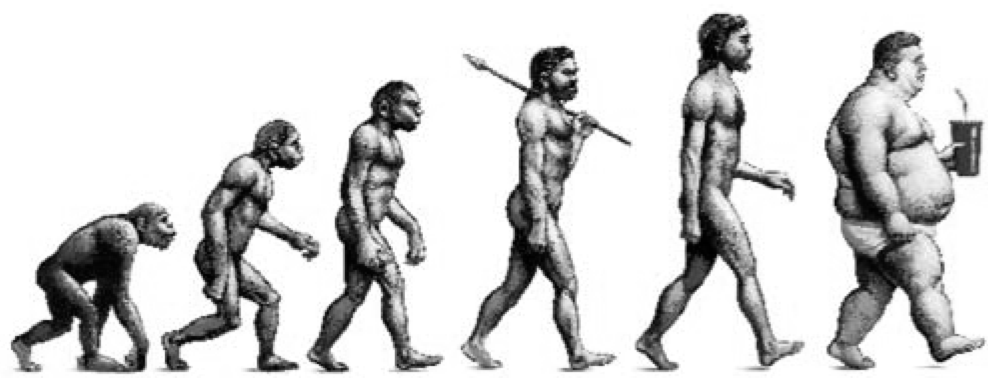
Please write your name VERY CLEARLY on each sheet and your student ID on the first page.

Write all your working on these question sheets. Use this cover page and the back page for extra working.

Watch the blackboard for corrections or clarifications during the exam.

This is a closed book exam. **No notes allowed. NO CALCULATORS!**

Standard notation for spherical coordinates is used throughout. Thus, for example, r is always the distance from the origin. As usual, take $V = 0$ at $r = \infty$ if and when appropriate.



 Name solutions

Section I.

1. [10] Show that $\nabla.(\mathbf{D}V) = V\nabla.\mathbf{D} + \mathbf{D}.\nabla V$ for any scalar field V and vector field $\mathbf{D}$.

$$\vec{\nabla}.(\vec{D}V) = \sum_i \frac{\partial}{\partial x_i}(\vec{D}V)_i = \sum_i \frac{\partial}{\partial x_i}(D_i V) \;\; \text{3pts} \;\; = \sum_i \left(\frac{\partial D_i}{\partial x_i} V + D_i \frac{\partial V_i}{\partial x_i} \right) \;\; \text{3pts}$$

$$= V \sum_i \frac{\partial D_i}{\partial x_i} + \sum_i D_i (\vec{\nabla} V)_i$$

$$= V\vec{\nabla}.\vec{D} + \vec{D}.\vec{\nabla} V \;\; \text{4pts}$$

4 pts for stating ∇ a Linear op

2. [10] Hence show that $\frac{1}{2}\int \rho_f V\, d^3x = \frac{1}{2}\int \mathbf{D}.\mathbf{E}\, d^3x$, where ρ_f is the free charge density, $\mathbf{D}$ is the displacement field, $\mathbf{E}$ is the electric field, and V is the electrostatic potential.

$\nabla.\vec{D} = \rho_f$ 2 pts

$$\int \rho_f V\, d^3x = \int \vec{\nabla}.\vec{D}\, V\, d^3x$$

$$= \int \left[\vec{\nabla}.(\vec{D}V) - \vec{D}.\vec{\nabla} V\right] d^3x \quad \text{Integrate by parts 3pts}$$

$$= \int \vec{\nabla}.(\vec{D}V)\, d^3x + \int \vec{D}.(-\vec{\nabla} V)\, d^3x \quad \text{2pts} \;\; -\nabla V = E$$

$$= \oint (\vec{D}V).d\vec{A} + \int \vec{D}.\vec{E}\, d^3x$$

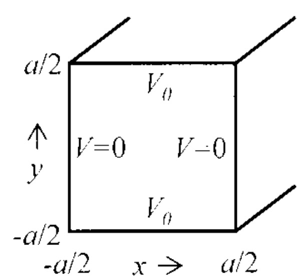
2pt surface term $= 0$, 1 pts fields $\to 0$ at $r = \infty$

by Gauss

this is zero for a sufficiently large surface because $|D| \sim \frac{1}{r^2}$, $V \sim \frac{1}{r}$ $\therefore$ integral $\sim \frac{1}{r}$ for large r

$\int dA \sim r^2$

Consider an infinitely long, square conducting pipe of internal side a. The horizontal faces, at $y = \pm a/2$, are held at potential V_0, while the vertical faces, at $x = \pm a/2$, are at $V = 0$. In the space on page 3, find an expression for the potential inside the pipe in the following way:



3. [20] By separation of variables, find the general solution $V(x,y)$ of $\nabla^2 V = 0$, and show by considering the boundary conditions and their symmetry that the potential in the pipe must be of the form

$$V(x,y) = \sum_{n=0}^{\infty} C_n \cos\left[(2n+1)\frac{\pi x}{a}\right] \cosh\left[(2n+1)\frac{\pi y}{a}\right].$$

4. [6] State the appropriate orthogonality relation for cosine functions.

5. [20] Derive an equation from the boundary condition at $y = a/2$, and use Fourier's technique to extract the coefficients C_n from it. Remember to extrapolate $V(x,a/2)$ over the period $2a$ appropriately.

Name solutions

$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}$ because V is indep of z

$= 0.$ Put $V(x,y) = X(x)Y(y)$ 3 pts

3 pts $\Rightarrow \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k^2$ = constant k can be real or imaginary

2 pts $\therefore X_k = a_k e^{ikx} + b_k e^{-ikx}$ are solutions

2 pts $Y_k = c_k e^{ky} + d_k e^{-ky}$

General solution: $V = \sum_k \left(a_k e^{ikx} + b_k e^{-ikx}\right)\left(c_k e^{ky} + d_k e^{-ky}\right)$

To match $V(\frac{a}{2}, y) = 0 = V(-\frac{a}{2}, y)$ we must have $X_k \propto \cos(2n+1)\frac{\pi x}{a}$

3 pts $\rightarrow \cosh kx$

3 pts $\longrightarrow k = \frac{(2n+1)\pi}{a}$ $n = 0, 1, \ldots$

To match $V(x, \frac{a}{2}) = V(x, -\frac{a}{2})$ we must have $Y_k \propto \cosh ky$ 3 pts

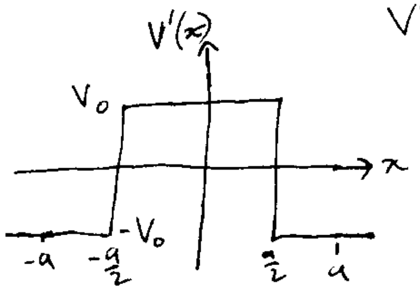
$\therefore V(x,y) = \sum_{n=0}^{\infty} C_n \cos\left[(2n+1)\frac{\pi x}{a}\right]\cosh\left[(2n+1)\frac{\pi y}{a}\right]$

Orthogonality: $\int_{-a}^{a} \cos\left[(2n+1)\frac{\pi x}{a}\right]\cos\left[(2m+1)\frac{\pi x}{a}\right]dx = a\,\delta_{mn}$ 1 pt 6 pt

Match b.c. at $y = \pm\frac{a}{2}$: $V(x, \frac{a}{2}) = \sum_{n=0}^{\infty} C_n \cos\left[(2n+1)\frac{\pi x}{a}\right]\cosh\left[(2n+1)\frac{\pi a}{2}\right]$

Need to extrapolate $V(x, \frac{a}{2})$ into a symmetric, periodic function

$V'(x)^* = V(-x) = V(x+2a)$ 3 pts



Use Fourier trick.

$\int_{-a}^{a} V'(x)\cos\frac{(2m+1)\pi x}{a}\,dx = C_m\, a\cosh\left[(2m+1)\frac{\pi a}{2}\right]$ 7 pts

$= 4\int_0^{a/2} V_0 \cos\frac{(2m+1)\pi x}{a}\,dx$ 2 pts factor of 2

integrate 4 pts

$= 4V_0\left[\frac{a}{(2m+1)\pi}\sin\left(\frac{2m+1}{a}\pi x\right)\right]_0^{a/2}$

$= \frac{4V_0 a}{(2m+1)\pi}\sin\frac{(2m+1)\pi}{2} = \frac{4V_0 a}{(2m+1)\pi}(-1)^m$ 1 pt ... $\frac{(2m-1)\pi}{2} = (-1)^n$

$\therefore C_n = \frac{4V_0}{(2n+1)\pi}(-1)^n\left\{\cosh\left[(2n+1)\frac{\pi a}{2}\right]\right\}^{-1}$ 3 pts

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Section II.

The potential in a region of space is initially given by $V_a(\mathbf{x}) = -E_a z$. A small conducting sphere of radius R held at potential V_0 is then placed in that region with its center at the origin.

6. [5] Write down the general solution, $V(r,\theta)$ of $\nabla^2 V = 0$ in spherical polar coordinates with azimuthal symmetry as an expansion in powers of r and Legendre polynomials $P_l(\cos\theta)$.

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-l-1} \right) P_l(\cos\theta)$$

1 pt 2 pts 2 pts

7. [18] Find the values of the coefficients in this expansion by matching the boundary conditions, and so show that the potential $V(\mathbf{x})$ for $r > R$ is equivalent to that of a charge Q and a dipole $\mathbf{p}$ at the origin added to $V_a(\mathbf{x})$. Give the expressions for Q and $\mathbf{p}$.

b.c.'s: (i) $V \to -E_a r\cos\theta$ as $r \to \infty$ (2 pts) $\therefore$ all $A_l = 0$ except $A_1 = -E_a$ (2 pts)

(ii) $V(R,\theta) = V_0$. (2 pts) $\therefore -E_a R\cos\theta + \sum_{l=0}^{\infty} B_l R^{-l-1} P_l(\cos\theta) = V_0$

coeffs of each power of $\cos\theta$ must match

$\therefore -E_a R + \dfrac{B_1}{R^2} = 0$ (3 pts) and $\dfrac{B_0}{R} = V_0$ (3 pts)

$\therefore B_1 = R^3 E_a$ and $B_0 = V_0 R$; all other B_l are zero

$\therefore$ we have $V(r,\theta) = -E_a r\cos\theta + \dfrac{V_0 R}{r} + \dfrac{R^3 E_a}{r^2}\cos\theta$

$$= -E_a z + \frac{Q}{4\pi\varepsilon_0 r} + \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

2 pts 2 pts

applied field — potential of charge $Q = 4\pi\varepsilon_0 V_0 R$ (2 pts) — potential of dipole $\vec{p} = 4\pi\varepsilon_0 R^3 E_a \hat{k}$ (2 pts)

8. [12] Determine the surface charge density $\sigma(\theta)$ on the sphere for nonzero V_0.

$\sigma = \varepsilon_0 E_n = -\varepsilon_0 \left.\dfrac{\partial V}{\partial r}\right|_R$ (3 pts)

3 pts 3 pts 3 pts

$$= -\varepsilon_0 \left(-\frac{V_0 R}{r^2} - E_a\cos\theta - 2\frac{R^3 E_a}{r^3}\cos\theta \right)_{r=R}$$

$$= \varepsilon_0 \left(\frac{V_0}{R} + 3E_a\cos\theta \right)$$

 Name solutions

9. [4] What is the quadrupole moment of this surface charge configuration?

0 — there's no quadrupole term in the potential.

4 pts

10. [5] Determine the polarizability α of the conducting sphere.

3 pts

$\vec{p} = \alpha \vec{E} = 4\pi\varepsilon_0 R^3 E_a \hat{k}$

$\therefore \alpha = 4\pi\varepsilon_0 R^3$ 2 pts for correct working

11. [~~6~~7] Sketch below the equipotentials (solid) and field lines (dashed) around the sphere for the case $V_0 = 0$. [Remember that $V_a(0) = 0$].

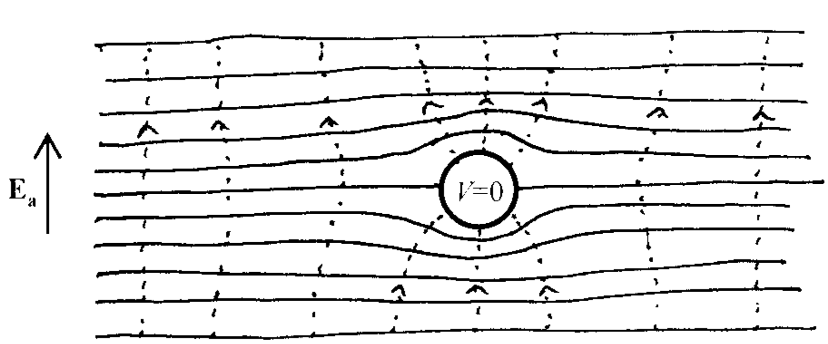
2 pts - uniform at large $\vec{r}$

1 pt - equipot $V=0$

2 pt - $\vec{E}$ normal to equipots & surface

1 pt - correct arrangement at sph. surface

1 - overall

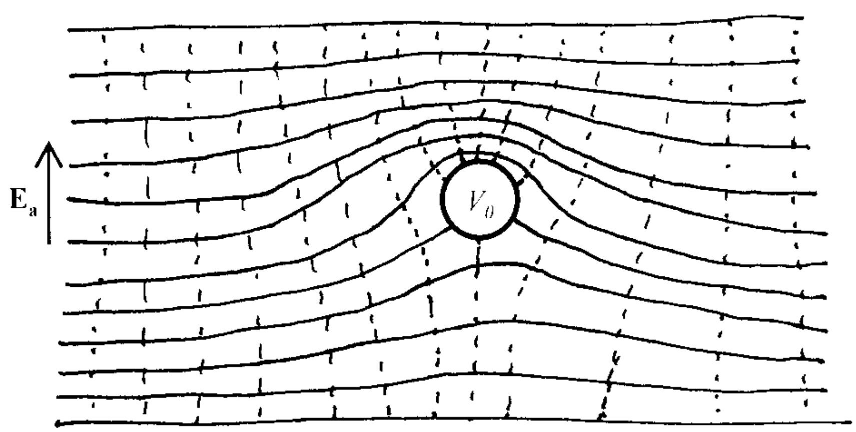


12. [6] Now do the same for positive V_0 (ie, positive Q).

2 pts - uniform at large $\vec{r}$

2 pts - equipots bunched on one side + expanded on other

1 pt - correct fields



13. [10] Which of the following is/are not true?

(a) $P_7(1) = 1$ TRUE 2 pts

(b) $P_6(x) = (231x^6 - 315x^4 + 105x^2 - 5)/16$ TRUE 2 pts

(c) $x^5 = [8P_5(x) + 28P_3(x) + 27P_1(x)]/63$ TRUE 2 pts

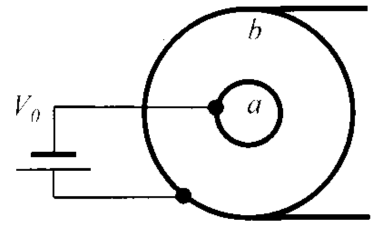
(d) $\int_0^\pi P_l(\cos\theta)P_m(\cos\theta)d\theta = 2\delta_{lm}/(2l+1)$ FALSE $\sin\theta$ is missing 2 pts

(e) $P_8(x) = 7/32\ P_{10}(x) - 16/21\ P_6(x) + 3/16\ P_4(x) - 8/3P_2(x) + 7/2$ FALSE 2 pts

you can't express one P_l in terms of the others.

 Name solutions

Section III.
A cable of length L consists of two coaxial cylindrical conductors, the inner having radius a and the outer radius $b << L$. One end of the cable is connected to a battery which maintains a potential V_0 on the inner conductor while the outer is grounded. The other end is open, and initially the space between the conductors is filled with air (whose dielectric properties may be neglected).



14. [14] Find the capacitance of the cable, using Gauss's law or otherwise.

Draw a Gaussian cylinder enclosing the inner conductor:

3 $\varepsilon_0 \oint E_n dA = Q_{enc}$ so $\varepsilon_0 E(r).2\pi r L = Q$

$$\therefore E(r) = \frac{Q}{2\pi r L \varepsilon_0} \quad 2$$

$$\therefore V_0 = \int_b^a -E(r)\,dr \;(2) = \int_a^b \frac{Q}{2\pi r \varepsilon_0 L} dr = \frac{Q}{2\pi\varepsilon_0 L} \ln\frac{b}{a} \quad 2$$

$$\therefore C = \frac{Q}{V_0} \;(2) = \frac{2\pi\varepsilon_0 L}{\ln b/a} \quad 3$$

15. [8] The cable is then immersed in oil with dielectric constant ε_r and the cable fills with oil. Find the electrical work done by the battery during this process. Neglect end effects and gravity.

New capacitance (3) is $C' = \varepsilon_r C$ (3) $= \dfrac{2\pi\varepsilon_r\varepsilon_0 L}{\ln b/a}$

Work done by battery $= V\Delta Q = V^2C' - V^2C = V^2(C'-C)$

$$= \frac{2\pi(\varepsilon_r - 1)\varepsilon_0 L V^2}{\ln b/a} \quad 2$$

16. [4] If you didn't neglect gravity, would the work done by the battery be different? Why?

No. (4) The same. The work is converted to electrostatic energy + gravitational energy + heat. More gravitational work done → less heat produced.

 Name solutions

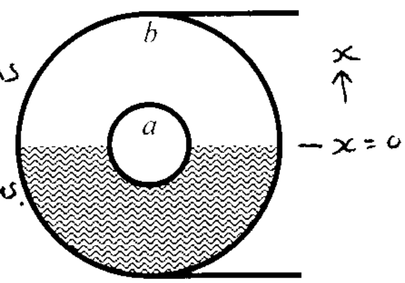
17. [10] The battery is now disconnected. Shortly after, the cable is removed from the oil, and the oil inside it quickly drains out. Neglecting charge leakage, what is the potential on the inner conductor immediately after it is drained?

Charge Q is constant. (3) $Q = C'V_0$ at the start (2)
$= C\,V_{final}$ at the end (2)

$$\therefore V_{final} = \frac{C'}{C} V_0 = \varepsilon_r V_0 \quad (3)$$

18. [10] Later, the cable is left lying horizontally on the ground, connected to the battery, exactly half filled with oil. Find **E** and **D** within the cable now.

(4)
$V(r)$ is unchanged. It still matches $\nabla^2 V = 0$ in the air and oil, the Dirichlet conditions at the surfaces of metal, and at the oil-air interface $D_n = 0$ and E_t is continuous.



$$\therefore E(r) = \frac{Q}{2\pi\varepsilon_0 r L} = \frac{CV_0}{2\pi\varepsilon_0 r L} = \frac{V_0}{r \ln b/a} \quad \text{radial only.}$$

$$\vec{E} = \frac{V_0}{r \ln b/a}\hat{r} \quad (2)$$

$$\vec{D} = \varepsilon_r \varepsilon_0 \vec{E} \quad \text{for } x < 0 \quad (2)$$
$$= \varepsilon_0 \vec{E} \quad \text{for } x > 0 \quad (2)$$

19. [12] What is the bound charge density in the oil … $\sigma_b = P_n = \vec{P}\cdot\hat{n}$ (2) $\quad \vec{P} = \chi\varepsilon_0\vec{E}$ (1) $\quad \chi = \varepsilon_r - 1$

(a) at the interface with the inner conductor, $\sigma_i = -P(r=a) = -\dfrac{\chi\varepsilon_0 V}{a \ln b/a}$ (2)

(b) at the interface with the outer conductor, $\sigma_o = P(r=b) = \dfrac{\chi\varepsilon_0 V}{b \ln b/a}$ (2)

(c) in the bulk, $\rho_b = \vec{\nabla}\cdot\vec{P} = 0$ (1)

(d) at the horizontal interface with the air? $\sigma_b = P_n = 0$ (1) as $\vec{P} \propto \hat{r}$ (1)

20. [8] Find the total energy stored using the expression $U = \frac{1}{2}\int \mathbf{D}.\mathbf{E}\, d^3x$.

$$U = \frac{1}{2}\int_{oil} \varepsilon_r \varepsilon_0 E^2 d^3x + \frac{1}{2}\int_{air} \varepsilon_0 E^2 d^3x \quad (2)$$

$$= \frac{1}{2}(\varepsilon_r + 1)\varepsilon_0 \int_{\substack{\text{half}\\ \text{cylinder}}} E^2 d^3x \quad (3)$$

$$= \frac{1}{2}(\varepsilon_r+1)\varepsilon_0 \int_a^b \left(\frac{V_0}{r \ln b/a}\right)^2 L.\pi r\, dr = \frac{\pi L(\varepsilon_r+1)\varepsilon_0 V_0^2}{2(\ln b/a)^2}\int_a^b \frac{dr}{r} = \frac{\pi L(\varepsilon_r+1)\varepsilon_0 V_0^2}{2\ln b/a} \quad (3)$$