

Electromagnetism, Physics 321
Autumn 2003

Final exam
Instructor: David Cobden

8.30 am, December 15, 2003

Do not turn this page until the buzzer goes at 8.30. You have 110 minutes. End on the buzzer at 10.20.

This exam has 20 questions, divided into three sections. Attempt all of them. Each section is worth 66 points. You have about 35 minutes for each section.

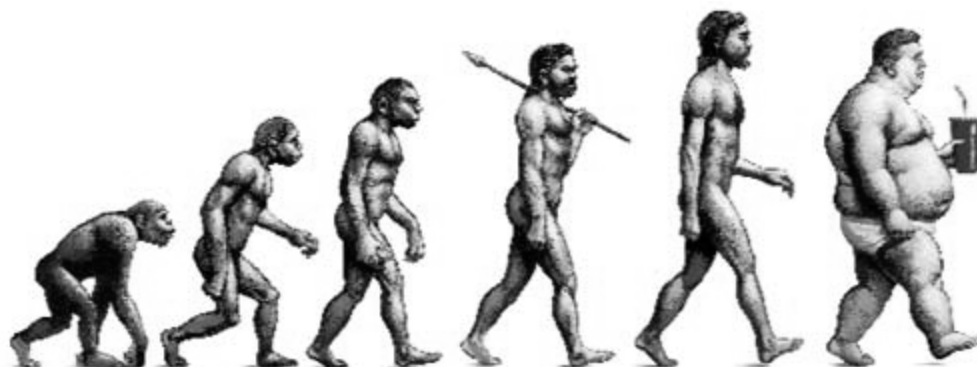
Please write your name **VERY CLEARLY** on each sheet and your student ID on the first page.

Write all your working on these question sheets. Use this cover page and the back page for extra working.

Watch the blackboard for corrections or clarifications during the exam.

This is a closed book exam. **No notes allowed. NO CALCULATORS!**

Standard notation for spherical coordinates is used throughout. Thus, for example, r is always the distance from the origin. As usual, take $V = 0$ at $r = \infty$ if and when appropriate.



Section I.

1. [10] Show that $\nabla \cdot (\mathbf{D}\nabla V) = \nabla \cdot \mathbf{D} V + \mathbf{D} \cdot \nabla V$ for any scalar field V and vector field \mathbf{D} .

$$\begin{aligned} \nabla \cdot (\mathbf{D}\nabla V) &= \sum_i \frac{\partial}{\partial x_i} (\mathbf{D}\nabla V)_i = \sum_i \frac{\partial}{\partial x_i} (D_i \nabla V)_i = \sum_i \left(\frac{\partial D_i}{\partial x_i} V + D_i \frac{\partial V}{\partial x_i} \right) \\ &= V \sum_i \frac{\partial D_i}{\partial x_i} + \sum_i D_i (\nabla V)_i \\ &= V \nabla \cdot \mathbf{D} + \mathbf{D} \cdot \nabla V \end{aligned}$$

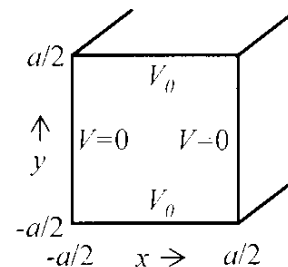
3 pts
4 pts for stating V is linear op

2. [10] Hence show that $\frac{1}{2} \int \rho_f V d^3x = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} d^3x$, where ρ_f is the free charge density, \mathbf{D} is the displacement field, \mathbf{E} is the electric field, and V is the electrostatic potential.

$$\begin{aligned} \int \rho_f V d^3x &= \int \nabla \cdot \mathbf{D} V d^3x && \nabla \cdot \mathbf{D} = \rho_f \quad 2 \text{ pts} \\ &= \int [\nabla \cdot (\mathbf{D}V) - \mathbf{D} \cdot \nabla V] d^3x && \text{Integrate by parts} \\ &= \int \nabla \cdot (\mathbf{D}V) d^3x + \int \mathbf{D} \cdot (-\nabla V) d^3x && 3 \text{ pts} \quad -\nabla V = \mathbf{E} \\ &= \oint (\mathbf{D}V) \cdot d\mathbf{A} + \int \mathbf{D} \cdot \mathbf{E} d^3x && \text{2 pts} \quad \text{apt surface term} = 0 \\ &&& \text{pts fields} \rightarrow 0 \text{ at } r = \infty \\ &&& \uparrow \text{ by Gauss} \end{aligned}$$

This is zero for a sufficiently large surface because $|\mathbf{D}| \sim \frac{1}{r^2}$, $V \sim \frac{1}{r}$ \therefore integral $\sim \frac{1}{r}$ for large r
 $\int dA \sim r^2$

Consider an infinitely long, square conducting pipe of internal side a . The horizontal faces, at $y = \pm a/2$, are held at potential V_0 , while the vertical faces, at $x = \pm a/2$, are at $V = 0$. In the space on page 3, find an expression for the potential inside the pipe in the following way:



3. [20] By separation of variables, find the general solution $V(x,y)$ of $\nabla^2 V = 0$, and show by considering the boundary conditions and their symmetry that the potential in the pipe must be of the form

$$V(x,y) = \sum_{n=0}^{\infty} C_n \cos \left[(2n+1) \frac{\pi x}{a} \right] \cosh \left[(2n+1) \frac{\pi y}{a} \right].$$

4. [6] State the appropriate orthogonality relation for cosine functions.

5. [20] Derive an equation from the boundary condition at $y = a/2$, and use Fourier's technique to extract the coefficients C_n from it. Remember to extrapolate $V(x, a/2)$ over the period $2a$ appropriately.

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \quad \text{because } V \text{ is indep of } z$$

$$= 0. \quad \text{Put } V(x, y) = X(x)Y(y) \quad 3 \text{ pts}$$

$$\begin{matrix} 3 \text{ pts} \\ \rightarrow \end{matrix} \frac{1}{X} \frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} = -k^2 = \text{constant } k \text{ can be real or imaginary}$$

$$2 \text{ pts } \therefore X_k = a_k e^{ikx} + b_k e^{-ikx} \quad \text{are solutions}$$

$$2 \text{ pts } Y_k = c_k e^{ky} + d_k e^{-ky}$$

$$\text{General solution: } V = \sum_k (a_k e^{ikx} + b_k e^{-ikx}) (c_k e^{ky} + d_k e^{-ky})$$

$$\text{To match } V\left(\frac{a}{2}, y\right) = 0 = V\left(-\frac{a}{2}, y\right) \quad \text{we must have } X_k \propto \cos\left(\frac{(2n+1)\pi x}{a}\right)$$

$$3 \text{ pts } \rightarrow \cos kx$$

$$3 \text{ pts } \rightarrow k = \frac{(2n+1)\pi}{a} \quad n=0, 1, \dots$$

$$\text{To match } V\left(x, \frac{a}{2}\right) = V\left(x, -\frac{a}{2}\right) \quad \text{we must have } Y_k \propto \cosh ky$$

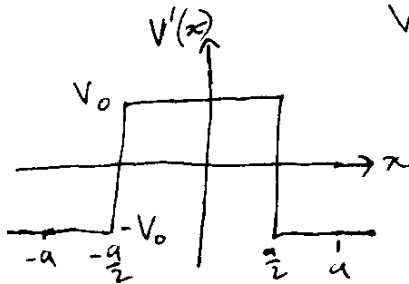
$$\therefore V(x, y) = \sum_{n=0}^{\infty} C_n \cos\left[\frac{(2n+1)\pi x}{a}\right] \cosh\left[\frac{(2n+1)\pi y}{a}\right] \quad 3 \text{ pts}$$

$$\text{Orthogonality: } \int_{-a}^a \cos\left[\frac{(2n+1)\pi x}{a}\right] \cos\left[\frac{(2m+1)\pi x}{a}\right] dx = a \delta_{mn} \quad 6 \text{ pts}$$

$$\text{Match bc. at } y = \pm \frac{a}{2}: \quad V\left(x, \frac{a}{2}\right) = \sum_{n=0}^{\infty} C_n \cos\left[\frac{(2n+1)\pi x}{a}\right] \cosh\left[\frac{(2n+1)\pi a}{2}\right]$$

Need to extrapolate $V\left(x, \frac{a}{2}\right)$ into a symmetric, periodic function

$$V'(x) = V(-x) = V(x+2a) \quad 3 \text{ pts}$$



Use Fourier trick.

$$\int_{-a}^a V'(x) \cos\left[\frac{(2m+1)\pi x}{a}\right] dx = C_m a \cos\left[\frac{(2m+1)\pi a}{2}\right] \quad 7 \text{ pts}$$

$$= 4 \int_0^{a/2} V_0 \cos\left[\frac{(2m+1)\pi x}{a}\right] dx \quad 2 \text{ pts} \quad \left. \begin{array}{l} \text{factor of 2} \\ \text{integrate} \\ 4 \text{ pts} \end{array} \right\}$$

$$= 4V_0 \left[\frac{a}{(2m+1)\pi} \sin\left[\frac{(2m+1)\pi x}{a}\right] \right]_0^{a/2}$$

$$= \frac{4V_0 a}{(2m+1)\pi} \sin\left[\frac{(2m+1)\pi}{2}\right] = \frac{4V_0 a}{(2m+1)\pi} (-1)^m \quad 1 \text{ pt} \quad \frac{\cos\frac{(2m+1)\pi}{2}}{(-1)^m}$$

$$\therefore C_n = \frac{4V_0}{(2n+1)\pi} (-1)^n \left\{ \cosh\left[\frac{(2n+1)\pi a}{2}\right] \right\}^{-1} \quad 3 \text{ pts}$$

9. [4] What is the quadrupole moment of this surface charge configuration?

0 - there's no quadrupole term in the potential.
4 pts

10. [5] Determine the polarizability α of the conducting sphere.

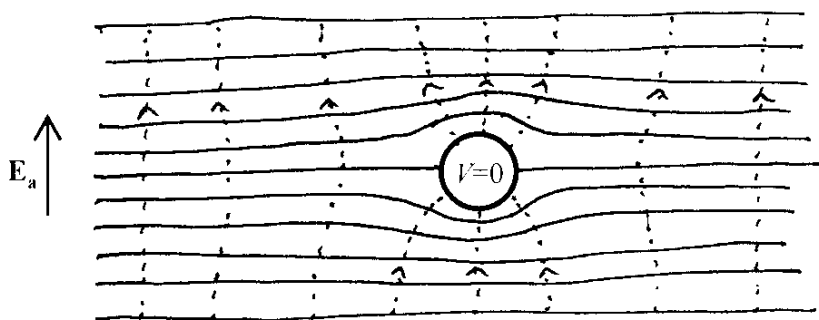
3 pts

$$\vec{p} = \alpha \vec{E} = 4\pi \epsilon_0 R^3 E_a \hat{k}$$

$$\therefore \alpha = 4\pi \epsilon_0 R^3$$
 2 pts for correct working

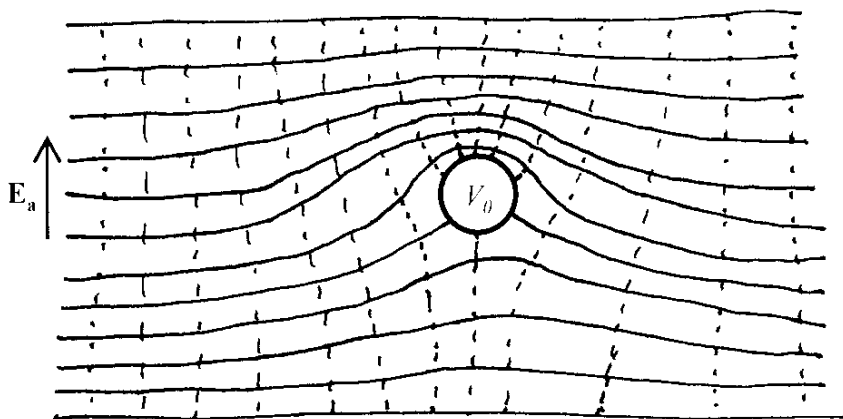
11. [7] Sketch below the equipotentials (solid) and field lines (dashed) around the sphere for the case $V_0 = 0$. [Remember that $V_a(0) = 0$].

2 pts - uniform at large r
 1 pt - equipot $V=0$
 2 pt - \vec{E} normal to equipots & surface
 1 pt - correct arrangement at sph. surface
 1 - overall



12. [6] Now do the same for positive V_0 (ie. positive Q).

2 pts - uniform at large r
 2 pts - equipots bunched on one side + expanded on other
 1 pt - correct fields



13. [10] Which of the following is/are not true?

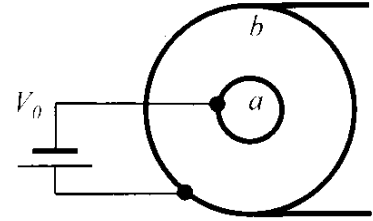
- (a) $P_7(1) = 1$ TRUE 2 pts
 - (b) $P_6(x) = (231x^6 - 315x^4 + 105x^2 - 5)/16$ TRUE 2 pts
 - (c) $x^5 = [8P_5(x) + 28P_3(x) + 27P_1(x)]/63$ TRUE 2 pts
 - (d) $\int_0^\pi P_l(\cos\theta)P_m(\cos\theta)d\theta = 2\delta_{lm}/(2l+1)$ FALSE $\sin\theta$ is missing 2 pts
 - (e) $P_8(x) = 7/32 P_{10}(x) - 16/21 P_6(x) + 3/16 P_4(x) - 8/3 P_2(x) + 7/2$ FALSE 2 pts
- you can't express one P_l in terms of the others.

Section III.

A cable of length L consists of two coaxial cylindrical conductors, the inner having radius a and the outer radius $b \ll L$. One end of the cable is connected to a battery which maintains a potential V_0 on the inner conductor while the outer is grounded. The other end is open, and initially the space between the conductors is filled with air (whose dielectric properties may be neglected).

14. [14] Find the capacitance of the cable, using Gauss's law or otherwise.

Draw a Gaussian cylinder enclosing the inner conductor:



$$\textcircled{3} \oint \epsilon_0 \mathbf{E} \cdot d\mathbf{A} = Q_{\text{enc}} \quad \text{so} \quad \epsilon_0 E(r) \cdot 2\pi r L = Q$$

$$\therefore E(r) = \frac{Q}{2\pi r L \epsilon_0} \quad \textcircled{2}$$

$$\therefore V_0 = \int_b^a -E(r) dr = \int_a^b \frac{Q}{2\pi r \epsilon_0 L} dr = \frac{Q}{2\pi \epsilon_0 L} \ln \frac{b}{a} \quad \textcircled{2}$$

$$\therefore C = \frac{Q}{V_0} = \frac{2\pi \epsilon_0 L}{\ln b/a} \quad \textcircled{3}$$

15. [8] The cable is then immersed in oil with dielectric constant ϵ_r and the cable fills with oil. Find the electrical work done by the battery during this process. Neglect end effects and gravity.

$$\text{New capacitance is } C' = \epsilon_r C = \frac{2\pi \epsilon_r \epsilon_0 L}{\ln b/a} \quad \textcircled{3}$$

$$\begin{aligned} \text{Work done by battery} &= V \Delta Q = V^2 C' - V^2 C = V^2 (C' - C) \\ &= \frac{2\pi (\epsilon_r - 1) \epsilon_0 L V^2}{\ln b/a} \quad \textcircled{2} \end{aligned}$$

16. [4] If you didn't neglect gravity, would the work done by the battery be different? Why?

No. The same. The work is converted to electrostatic energy + gravitational energy + heat.
 $\textcircled{4}$ More gravitational work done \rightarrow less heat produced.

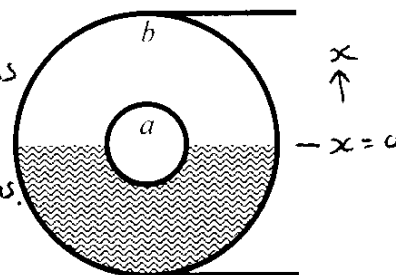
17. [10] The battery is now disconnected. Shortly after, the cable is removed from the oil, and the oil inside it quickly drains out. Neglecting charge leakage, what is the potential on the inner conductor immediately after it is drained?

Charge Q is constant. $Q = C'V_0$ at the start (2)
 (3) $= CV_{\text{final}}$ at the end (2)

$$\therefore V_{\text{final}} = \frac{C'}{C} V_0 = \epsilon_r V_0 \quad (3)$$

18. [10] Later, the cable is left lying horizontally on the ground, connected to the battery, exactly half filled with oil. Find E and D within the cable now.

$V(r)$ is unchanged. It still matches $\nabla^2 V = 0$ in the air and oil, the Dirichlet conditions at the surfaces of metal, and at the oil-air interface $D_n = 0$ and E_t is continuous.



$$\therefore E(r) = \frac{Q}{2\pi\epsilon_r r L} = \frac{CV_0}{2\pi\epsilon_r r L} = \frac{V_0}{r \ln(b/a)}$$

radial only.

$$\vec{E} = \frac{V_0}{r \ln(b/a)} \hat{r} \quad (2)$$

$$\vec{D} = \epsilon_r \epsilon_0 \vec{E} \quad \text{for } x < 0 \quad (2)$$

$$= \epsilon_0 \vec{E} \quad \text{for } x > 0 \quad (2)$$

$$\sigma_b = P_n = \vec{P} \cdot \hat{n} \quad (2)$$

$$\vec{P} = \chi \epsilon_0 \vec{E} \quad (1)$$

$$\chi = \epsilon_r - 1$$

19. [12] What is the bound charge density in the oil ...

(a) at the interface with the inner conductor,

$$\sigma'_i = -P(r=a) = -\chi \epsilon_0 V \quad (2)$$

(b) at the interface with the outer conductor,

$$\sigma'_o = P(r=b) = \chi \epsilon_0 V \frac{a \ln(b/a)}{b \ln(b/a)} \quad (2)$$

(c) in the bulk,

$$\rho_b = \nabla \cdot \vec{P} = 0 \quad (1)$$

(d) at the horizontal interface with the air?

$$\sigma_b = P_n = 0 \quad \text{as } \vec{P} \propto \hat{r} \quad (1)$$

20. [8] Find the total energy stored using the expression $U = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} d^3x$.

$$U = \frac{1}{2} \int_{\text{oil}} \epsilon_r \epsilon_0 E^2 d^3x + \frac{1}{2} \int_{\text{air}} \epsilon_0 E^2 d^3x \quad (2)$$

$$= \frac{1}{2} (\epsilon_r + 1) \epsilon_0 \int E^2 d^3x$$

half cylinder (3)

$$= \frac{1}{2} (\epsilon_r + 1) \epsilon_0 \int_a^b \left(\frac{V_0}{r \ln(b/a)} \right)^2 L \cdot \pi r dr = \frac{\pi L (\epsilon_r + 1) \epsilon_0 V_0^2}{2 (\ln(b/a))^2} \int_a^b \frac{dr}{r} = \frac{\pi L (\epsilon_r + 1) \epsilon_0 V_0^2}{2 \ln(b/a)} \quad (3)$$