Name Solutions

Electrodynamics, Physics 321 **First midterm** **8.20 am, November 1, 2004**
Autumn 2004 Instructor: David Cobden

You have 60 minutes. End on the buzzer at 9.20. Answer all 13 questions.

Write your name on every page and your ID on the first page.

Write all your working on these question sheets. Use this cover page for extra working (you might get credit for it.)

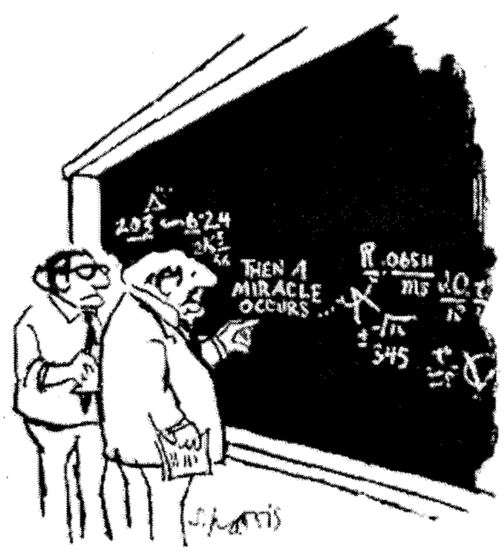
It is important to show your calculation or derivation. You won't get full marks just for stating the correct answer if you don't show how you get it.

Watch the blackboard for corrections or clarifications during the exam.

This is a **closed book** exam. **No notes allowed. No calculators!**

This exam contains 100 points, divided into four sections (one per page) each worth 25 points. Be sure to attempt all the questions, and ***spend no more than 15 minutes on each section before moving on***!

Do not turn this page until I say 'go'!



"I think you should be more explicit here in step two."

 Name solutions

I. General theorem of electrostatics:

1. [8] State carefully *both* the general differential relationship *and* the integral relationship between the charge density $\rho(\mathbf{r})$ and the electric potential $V(\mathbf{r})$ over all space.

$$\nabla^2 V(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon_0} \qquad (i)$$

$$V(\vec{r}) = \int_{\text{all space}} \frac{\rho(\vec{r}\,')}{4\pi\varepsilon_0 |\vec{r}-\vec{r}\,'|} d^3r' \qquad (ii)$$

2. [5] Give expressions for the charge density $\rho_1(\mathbf{r})$ and the potential $V_1(\mathbf{r})$ corresponding to a point charge q located at the origin.

$$\rho_1(\vec{r}) = q\,\delta^3(\vec{r}) \qquad (1)$$

$$V_1(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r} \qquad (2)$$

3. [12] Illustrate the validity of both the relationships in question 1 using the results for the point charge in question 2. You may use the result that $\nabla.(\mathbf{r}/r^3) = 4\pi\delta^3(\mathbf{r})$.

Put (2) into LHS of (i):

$$\nabla^2 V_1(\vec{r}) = \nabla^2\left(\frac{q}{4\pi\varepsilon_0 r}\right) = \frac{q}{4\pi\varepsilon_0}\nabla^2\left(\frac{1}{r}\right) = \frac{q}{4\pi\varepsilon_0}\vec{\nabla}.\left[\vec{\nabla}\left(\frac{1}{r}\right)\right]$$

$$\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\vec{\nabla}r = -\frac{\vec{r}}{r^3}$$

$$= \frac{-q}{4\pi\varepsilon_0}\vec{\nabla}.\left(\frac{\vec{r}}{r^3}\right)$$

$$= \frac{-q\,\delta^3(\vec{r})}{\varepsilon_0} = \frac{-\rho_1(\vec{r})}{\varepsilon_0}$$

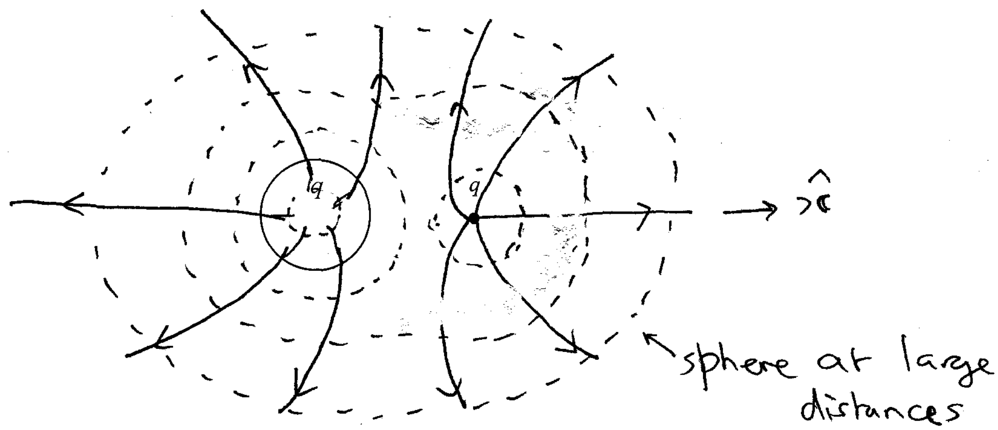
Put (1) into RHS of (ii):

$$\int \frac{\rho_1(\vec{r}\,')}{4\pi\varepsilon_0|\vec{r}-\vec{r}\,'|} d^3r' = \int \frac{q\,\delta^3(\vec{r}\,')}{4\pi\varepsilon_0|\vec{r}-\vec{r}\,'|} d^3r' = \frac{q}{4\pi\varepsilon_0|\vec{r}-0|} = \frac{q}{4\pi\varepsilon_0 r} = V_1(\vec{r})$$

 Name ~~solutions~~

II. A *nonconducting* sphere of radius R containing a fixed uniform positive charge density ρ_0 and total charge $q = (4/3)\pi R^3 \rho_0$ is centered at the origin. A particle with the same charge q is initially located at a distance $3R$ from the origin.

4. [6] Sketch the electric field lines (solid lines) and equipotentials (dashed lines) in this configuration.



5. [7] Find the force on the particle. Take care to specify the direction.

$\vec{F}_q = \vec{E}_{sphere}\, q$ Get $\vec{E}_{sphere}$ from Gauss by symmetry

outside sphere, $E_{sphere}(\vec{r}) = \dfrac{q\,\hat{r}}{4\pi\varepsilon_0 r^2}$ same as charge q at origin

$$\therefore \vec{F}_q = \frac{q^2\,\hat{x}}{4\pi\varepsilon_0 (3R)^2} = \frac{q^2\,\hat{x}}{36\pi\varepsilon_0 R^2}$$

6. [12] Find the change in potential energy of the particle if is moved from its initial position to the origin (the center of the sphere).

inside sphere, $\oint \vec{E}\cdot d\vec{S} = 4\pi r^2 E(r)$ by Gauss $= Q_{enc} = q$

$$\therefore 4\pi r^2 E(r) = \frac{4}{3}\pi r^3 \frac{\rho_0}{\varepsilon_0} = \frac{q\,r^3}{\varepsilon_0 R^3} \quad \text{so } E(\vec{r}) = \frac{q\,r}{4\pi\varepsilon_0 R^3}$$

$$W = -\int_{start}^{0} \vec{F}_q \cdot d\vec{l} = -\int_{3R}^{R} q\,E_{outside}(r)\,dr - \int_{R}^{0} q\,E_{inside}(r)\,dr$$

$$= -\int_{3R}^{R} \frac{q^2}{4\pi\varepsilon_0 r^2}\,dr - \int_{R}^{0} \frac{q^2 r}{4\pi\varepsilon_0 R^3}\,dr$$

$$= \frac{q^2}{4\pi\varepsilon_0}\left\{ -\left[\frac{-1}{r}\right]_{3R}^{R} - \left[\frac{r^2}{2R^3}\right]_{R}^{0} \right\} \qquad = \frac{7q^2}{24\pi\varepsilon_0 R}$$

$$= \frac{q^2}{4\pi\varepsilon_0}\left\{ \frac{1}{R} - \frac{1}{3R} - 0 + \frac{R^2}{2R^3} \right\} = \frac{q^2}{4\pi\varepsilon_0 R}\left(1 - \frac{1}{3} + \frac{1}{2}\right)$$

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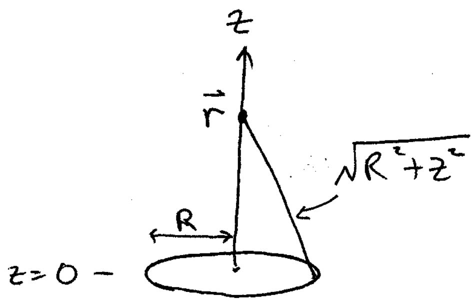
III. A circle of radius R has a fixed line charge density λ along its circumference. Take it to lie in the plane $z = 0$ centered on the z axis.

7. [15] Find the electrostatic potential $V(z)$ along the axis. Indicate your choice of a reference. Show that its form in the limit of large z is what you would expect.



All charge on loop is equidistant from point $\vec{r}$ on the axis so

$$|\vec{r}-\vec{r}\,'| = \sqrt{R^2+z^2} \qquad r = z$$

$$\therefore V(z)_{\text{on axis}} = \frac{2\pi R\lambda}{4\pi\varepsilon_0\sqrt{R^2+z^2}}$$

total charge on loop. $Q = 2\pi R\lambda$

For large z, $V(z) \doteq \dfrac{2\pi R\lambda}{4\pi\varepsilon_0 z} = \dfrac{Q}{4\pi\varepsilon_0 z} = \dfrac{Q}{4\pi\varepsilon_0 r}$ ✓

looks like point charge Q.

8. [10] A particle of charge q and mass m moves freely along the axis, subject only to electrostatic forces, and has velocity v_0 just when it passes through the center of the circle (ie, the origin). Find an expression for its velocity as a function of z.

Cons of energy: $qV(z) + \frac{1}{2}mv^2 = \text{const.}$ $\qquad v$ = speed along z-axis

At $z=0$, $v = v_0$

$$\therefore q\left[V(z) - V(0)\right] + \tfrac{1}{2}m\left(v^2 - v_0^2\right) = 0$$

$$V(0) = \frac{2\pi R\lambda}{4\pi\varepsilon_0 R} = \frac{\lambda}{2\varepsilon_0}$$

$$\therefore v^2 = v_0^2 + \frac{2q}{m}\left(\frac{\lambda}{2\varepsilon_0} - \frac{R\lambda}{2\varepsilon_0\sqrt{R^2+z^2}}\right)$$

$$= v_0^2 + \frac{q\lambda}{m\varepsilon_0}\left(1 - \frac{R}{\sqrt{R^2+z^2}}\right)$$

check: $v = v_0$ at $z = 0$

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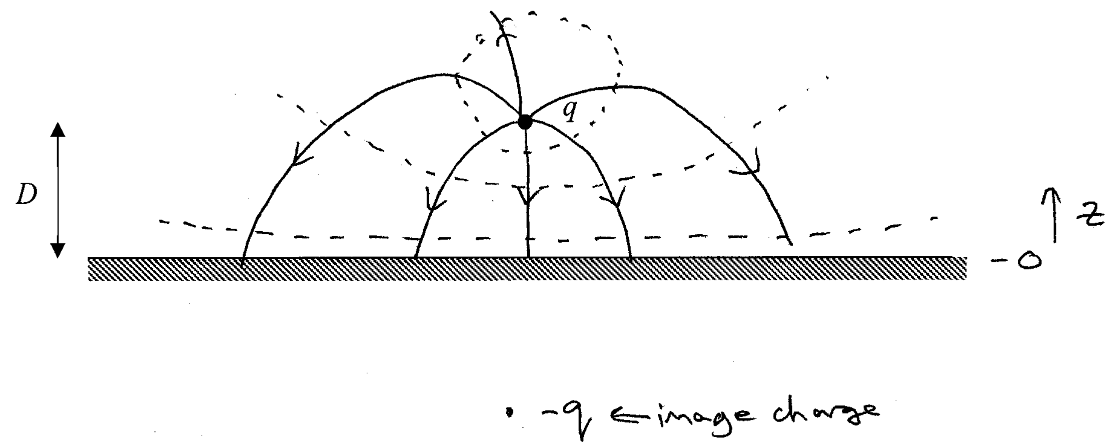
IV. A particle with positive charge q is located a distance D from the flat surface of a large piece of metal which is grounded (ie, $V = 0$ on the metal).

9. [6] Sketch the electric field (solid lines) and equipotential surfaces (dashed).



10. [3] State the relationship between the electric field and the surface charge density σ on the metal.

$\vec{E}_{surface} = \frac{\sigma}{\varepsilon_0}\hat{n}$ $\hat{n}$ = normal vector

11. [7] What is the force exerted on the particle? Indicate the key principles at work.

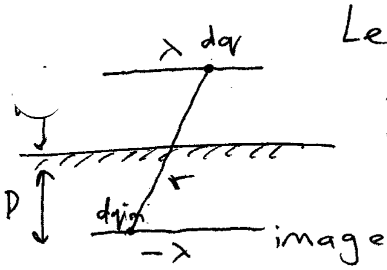
Match b.c.s on Laplace's equation if we have an image charge $-q$ at $z = -D$

$$\vec{F} = q \times (\text{field of image charge})$$
$$= q \times \left[\frac{-q\hat{z}}{4\pi\varepsilon_0(2D)^2}\right] = \frac{-q\hat{z}}{16\pi\varepsilon_0 D^2}$$

12. [4] By how much does the force on the particle change if the potential of the metal is increased to V_0 by connecting it to a battery?

Zero.

13. [5] If the particle were replaced by a charged wire, oriented parallel to the surface, having the same total charge q at the same distance D from the surface, would the force on the wire be greater than, equal to, or less than the force on the point particle? (A sketch might help you decide).



Less. Most bits of the image charge are further away from each bit of the real charge than they would be for a point charge.

$r \geq 2D$ for any pair dq and dq_{im}

End