Electrodynamics, Physics 321 **Second midterm** **8.20 am, November 19, 2004**
Autumn 2004 Instructor: David Cobden

You have 60 minutes. End on the buzzer at 9.20. Answer all **11** questions.

Write your name on every page and your ID on the first page.

Write all your working on these question sheets. Use this cover page for extra working (you might get credit for it.)

It is important to show your calculation or derivation. You won't get full marks just for stating the correct answer if you don't show how you get it.

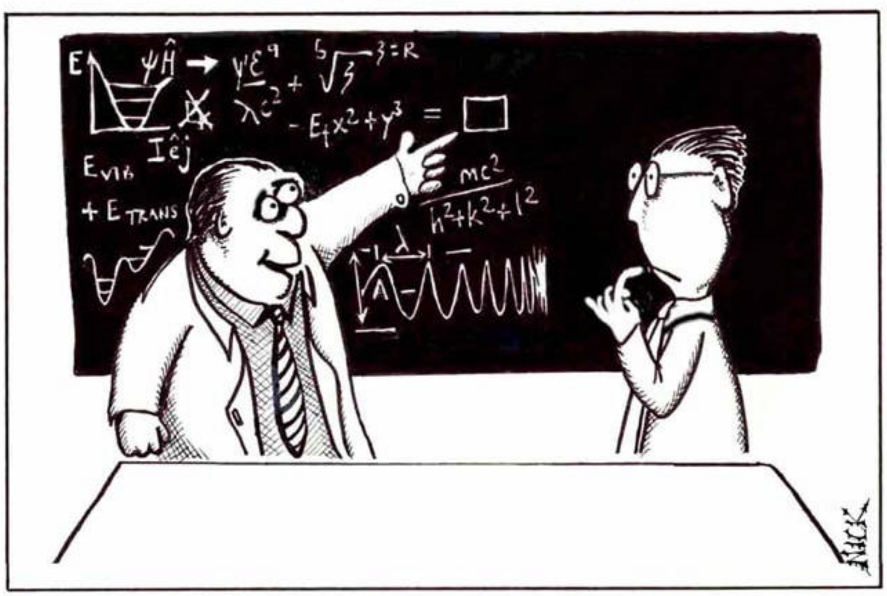
Watch the blackboard for corrections or clarifications during the exam.

This is a **closed book** exam. **No notes allowed. No calculators!**

Do not turn this page until I say GO!

$P_0(x) = 1$ $P_1(x) = x$ $P_2(x) = (3x^2-1)/2$ $P_3(x) = (5x^3 - 3x)/2$

Cartoon Physicists develop a grand unified theory:



"If my calculations are correct, not only must we always wear white lab coats, but the boundaries of our existence are defined solely by what is allowed to occur within the confines of a small two-dimensional box..!"

 Name solutions

I. A neutral conducting sphere of radius R is placed in a uniform external electric field $\mathbf{E_0} = E_0\,\hat{\mathbf{z}}$.

1. [6] State the uniqueness theorem for Laplace's equation, invoking Dirichlet and Neumann boundary conditions, which you should define.

For a region $\mathcal{V}$ bounded by S, if either V (Dirichlet) or $\frac{\partial V}{\partial n}$ (Neumann) is specified everywhere on S then there is only one unique solution for $V(\vec{r})$ within $\mathcal{V}$

2. [6] Give the eigenfunction expansion of the solution of Laplace's equation in spherical coordinates with azimuthal symmetry.

$$V(\vec{r}) = \sum_l \left(a_l r^l + \frac{b_l}{r^{l+1}}\right) P_l(\cos\theta)$$

3. [14] By applying the appropriate boundary conditions, use this expansion to find the potential $V(\mathbf{r})$ outside the sphere.

b.c.'s: (i) $V = 0$ at $r = R$ $\rightarrow$ $0 = \sum_l \left(a_l R^l + \frac{B_l}{R^{l+1}}\right) P_l(\cos\theta)$

because the P_l's are orthogonal, $a_l R^l = -\frac{B_l}{R^{l+1}}$ for all l

$$\therefore V(r,\theta) = \sum_{l=0}^{\infty} a_l \left(r^l - \frac{R^{2l+1}}{r^{l+1}}\right) P_l(\cos\theta)$$

(ii) $V \rightarrow -E_0 r\cos\theta$ for large r.

$$\therefore -E_0 r\cos\theta = \sum_l a_l r^l P_l(\cos\theta)$$

Equating coeffs of r, $a_1 P_1(\cos\theta) = -E_0\cos\theta$

$\therefore A_1 = -E_0$

$A_l = 0$ for $l \neq 1$

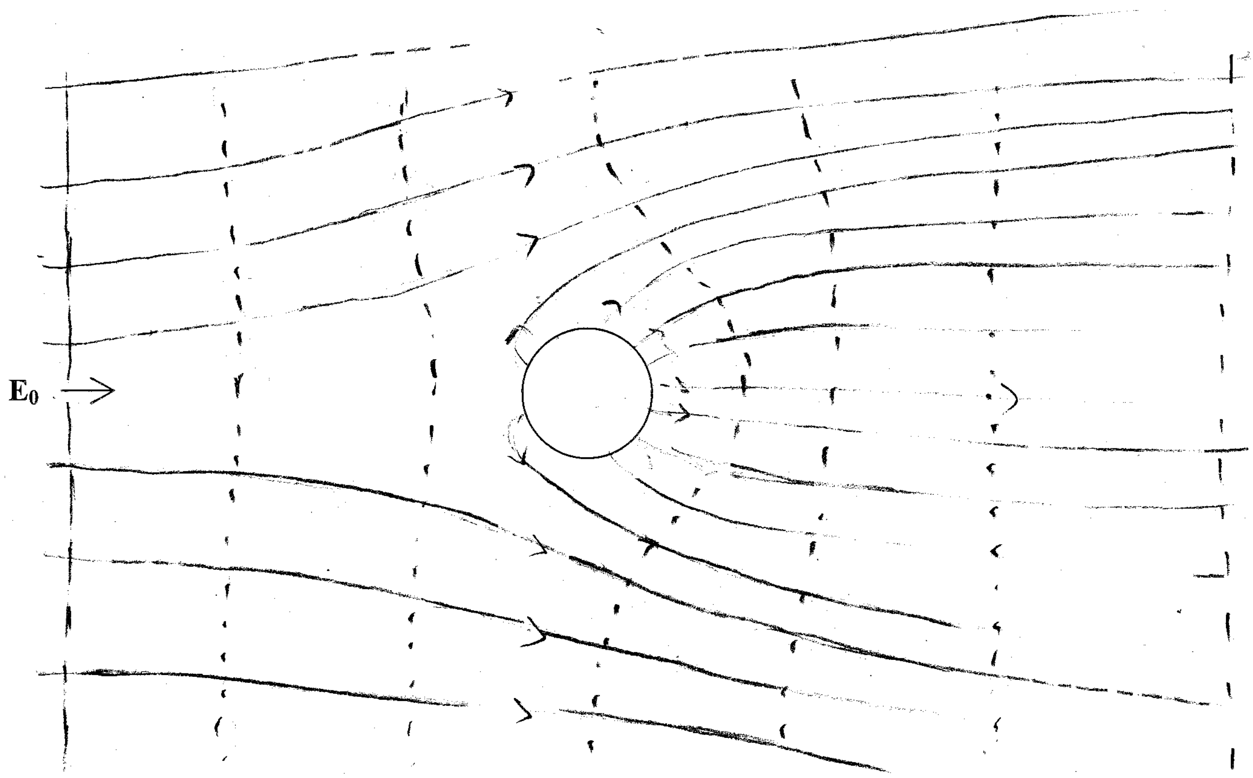
$$\therefore V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta$$

 Name solutions

4. [6] A charge Q is added to the sphere. What is the potential $V(\mathbf{r})$ outside the sphere now?

By superposition, $$V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta + \frac{Q}{4\pi\varepsilon_0 r}$$

5. [8] Sketch the electric field lines (solid lines) and equipotentials (dashed) around the sphere in the case $Q = 20\varepsilon_0 R^2 E_0$.



6. [10] Find the charge density on the surface of the sphere.

$$\sigma = \varepsilon_0 E_n$$

$$= -\varepsilon_0 \left.\frac{\partial V}{\partial r}\right|_R = \varepsilon_0\left[E_0\left(1 + \frac{2R^3}{R^3}\right)\cos\theta + \frac{Q}{4\pi\varepsilon_0 R^2}\right]$$

$$= \varepsilon_0 E_0 3\cos\theta + \frac{Q}{4\pi R^2}$$

 Name solutions

II. Mainly dipoles.

7. [12] A positive point charge q and a negative point charge $-q$ are separated by a distance b. A neutral conducting sphere of radius $R < b/2$ is placed midway between them. By considering image charges, find the induced dipole moment in the sphere.

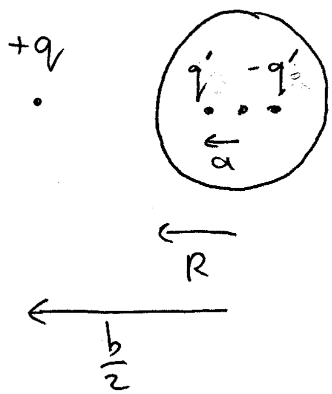


Image of $+q$ is q' as shown and q'' at origin.

Image of $-q$ is $-q'$ as shown and $-q''$ at origin.

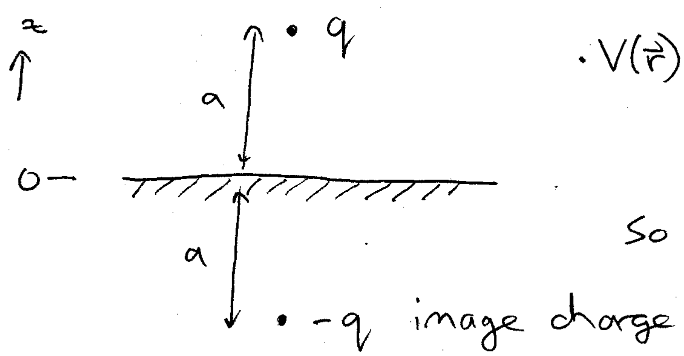
Net image charges, by superposition: q' and $-q'$ as shown; nothing at origin.

Dipole moment of sphere $p = |q' \times 2a|$ $\quad q' = \frac{-R}{(\frac{b}{2})} q \quad a = R^2/(b/2)$

$$\therefore \; p = \frac{R}{(b/2)} q \times \frac{2R^2}{(b/2)} = \frac{8R^3}{b^2} q$$

8. [8] A point charge q is placed a distance a from the conducting plane $z = 0$. What is the dominant term in the potential $V(\mathbf{r})$ for $r >> a$ and $z > 0$?

Combination of charge & image is physical dipole



$$V(\vec{r}) = \underset{\text{monopole}}{0} + \frac{\vec{p} \cdot \hat{r}}{4\pi\varepsilon_0 r^2} + \text{quadrupole} + \dots$$

$$\vec{p} = 2aq\,\hat{z}$$

So for large r,

$$V(\vec{r}) \to \frac{2aq\,\hat{z}\cdot\hat{r}}{4\pi\varepsilon_0 r^2} = \frac{aq\,z}{2\pi\varepsilon_0 r^3}$$

9. [6] Show that the Legendre polynomials $P_1(x)$ and $P_3(x)$ are orthogonal.

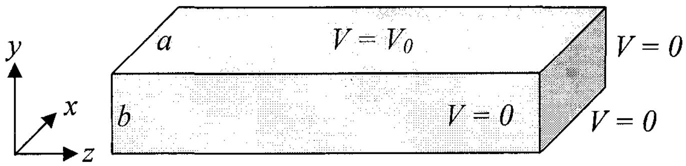
$$\int_{-1}^{1} P_1(x) P_3(x)\,dx = \int_{-1}^{1} x \cdot \left(\frac{5x^3 - 3x}{2}\right) dx$$

$$= \frac{1}{2} \int_{-1}^{1} (5x^4 - 3x^2)\,dx = \frac{1}{2}\left[x^5 - x^3\right]_{-1}^{1} = 0$$

 Name solutions

III. An infinite, straight tube with rectangular cross section of area ab is defined by the relevant parts of the planes $x = 0$, $x = a$, $y = 0$ and $y = b$. The side $y = b$ is held at potential $V = V_0$ whilst the other sides are all held at $V = 0$.

The potential inside the tube can be written in the form $V(x,y,z) = \sum_n A_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}$.



10. [12] Show that this solution matches all requirements except for one of the boundary conditions.

$$\nabla^2 V = \sum_n A_n \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}$$

$$= \sum_n A_n \left(-\frac{n^2\pi^2}{a^2} + \frac{n^2\pi^2}{a^2} + 0\right) \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a} = 0 \quad \checkmark \text{ obeys Laplace}$$

$$\frac{\partial V}{\partial z} = 0 \quad \checkmark \text{ obeys translational symmetry along } z$$

$$V(0,y,z) = \sum_n A_n \sin 0 \sinh\frac{n\pi y}{a} = 0 \quad \checkmark$$

$$V(a,y,z) = \sum_n A_n \sin n\pi \sinh\frac{n\pi y}{a} = 0 \quad \checkmark$$

$$V(x,0,z) = \sum_n A_n \sin\frac{n\pi x}{a} \sinh 0 = 0 \quad \checkmark$$

11. [12] Find the coefficients A_n from that remaining boundary condition.

$$V(x,b,z) = V_0 \text{ for } 0 < x < a \quad = \sum_n A_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi b}{a}$$

Fourier trick: $\rightarrow \int_0^a V_0 \sin\frac{n\pi x}{a}\,dx = \frac{a}{2} A_n \sinh\frac{n\pi b}{a}$

$$= V_0 \left[-\frac{a}{n\pi} \cos\frac{n\pi x}{a}\right]_0^a$$

$$= \frac{V_0 a}{n\pi}(1 - \cos n\pi)$$

$$= \frac{2V_0 a}{n\pi} \quad n \text{ odd} \quad \text{or zero for } n \text{ even}$$

$$\therefore A_n = \frac{4V_0}{n\pi}\left(\sinh\frac{n\pi b}{a}\right)^{-1} \quad \text{for } n \text{ odd}$$

$$= 0 \quad \text{for } n \text{ even.}$$