Name Solutions

Electrodynamics, Physics 321
Autumn 2004

Final exam
Instructor: David Cobden

8.20 am, December 14, 2004

You have 120 minutes. End on the buzzer at 10.20. Answer all 16 questions. They are divided into four sections each worth 50 points, 1 section per sheet of paper. So aim to spend half an hour on each sheet.

Write your name on every page and your ID on the first page.

Write all your working on these question sheets. Use this cover page for extra working (you might get credit for it.)

It is important to show your calculation or derivation. You won't get full marks just for stating the correct answer if you don't show how you get it.

Watch the blackboard for corrections or clarifications during the exam.

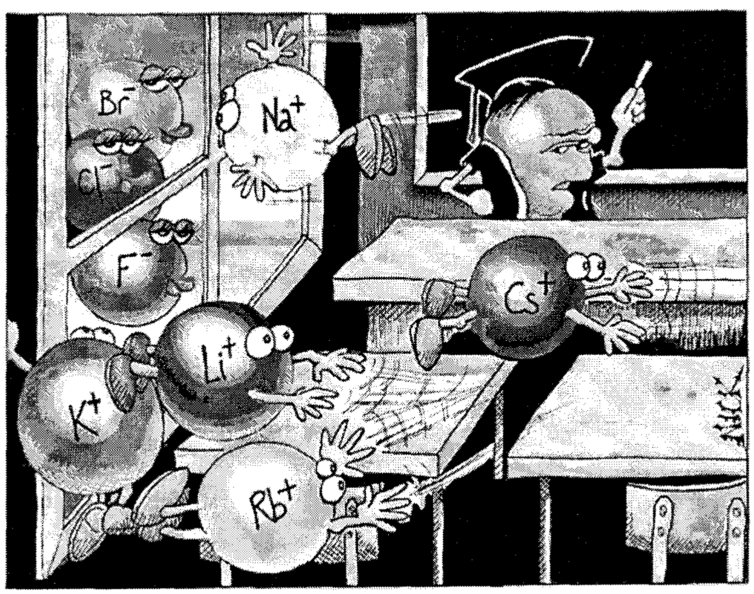
This is a **closed book** exam. **No notes allowed. No calculators!**

Do not turn this page until I say GO!

$P_l(1) = 1$

$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = (3x^2-1)/2 \qquad P_3(x) = (5x^3 - 3x)/2 \qquad P_4(x) = (35x^4 - 30x^2 + 3)/8$

$$-\nabla\left(\frac{\mathbf{p}.\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2}\right) = \frac{3(\mathbf{p}.\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{4\pi\varepsilon_0 r^3} \qquad \varepsilon_0 \approx 10^{-11}\ \text{F/m}$$



"Perhaps one of you gentlemen would mind telling me just what it is outside the window that you find so attractive..?"

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I. *The questions in this section ask for estimates. Show your reasoning, explain clearly assumptions or approximations made, and do all calculations algebraically first.*

A spaceship has a net charge $Q = +10^{-5}$ C on it when it leaves the earth's surface. Assume the earth is a conductor left with a net charge $-Q$, ie, the earth-spaceship system is neutral. [$R_{earth} \approx 6{,}000$ km.]

1. [10] *Estimate* the resulting change in the electrostatic potential of the earth between the point when the spaceship departs and when it leaves the solar system.

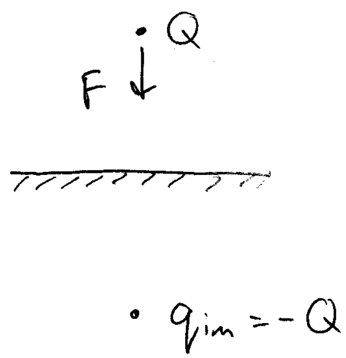
The result is to remove charge Q from a conducting sphere:

$$\Delta V = \frac{\Delta Q_{earth}}{C_{earth}} = \frac{-10^{-5}\text{ Coulombs}}{4\pi\varepsilon_0 R_{earth}}$$

$$\simeq \frac{-10^{-5}}{12\times10^{-11}\times 6\times10^{6}}\text{ Volts} = \frac{-10^{11-5-6}}{12\times 6}\text{ V} \simeq \frac{-1}{100}\text{ V}$$

$$\simeq -10\text{ mV}$$

2. [7] Estimate the electrostatic force on the spaceship when it is a distance $d = 1$ km from the ground.

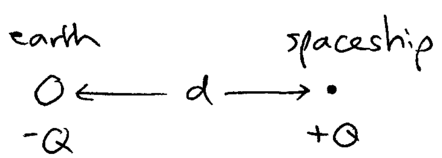


$\simeq$ image in conducting plane

$$F \simeq \frac{Q^2}{4\pi\varepsilon_0 (2d)^2} \simeq \frac{(10^{-5})^2}{12\times10^{-11}\times(2\times10^{3})^2}\text{ N}$$

$$= \frac{10^{-10+11-6}}{12\times 4}\text{ N} \simeq \frac{10^{-5}}{50}\text{ N} \simeq 2\times10^{-7}\text{ N}$$

3. [7] Estimate the force when $d = 10^6$ km.



Now $$F \simeq \frac{Q^2}{4\pi\varepsilon_0 d^2} \simeq \frac{(10^{-5})^2}{12\times10^{-11}\times(10^{9})^2}\text{ N}$$

$$= \frac{10^{-10+11-18}}{12}\text{ N}$$

$$\simeq 10^{-18}\text{ N}$$

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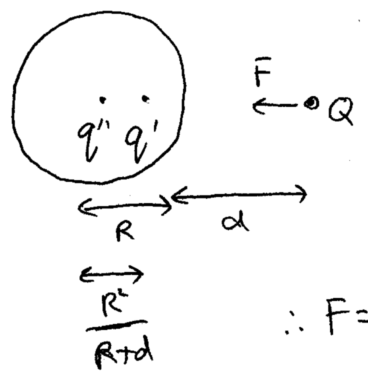
4. [11] Estimate the force when $d = 10^4$ km (ie, similar to R_{earth}).

Now we can't use either of the previous approximations.
We have image-in-sphere problem.

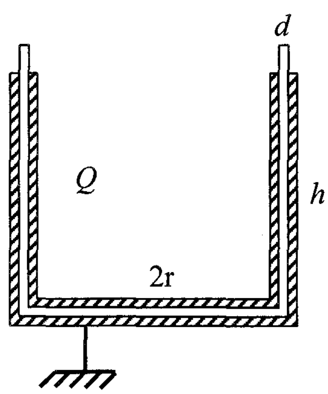


$$F = \frac{-1}{4\pi\varepsilon_0}\left[\frac{Qq''}{(R+d)^2} + \frac{Qq'}{\left(R+d-\frac{R^2}{R+d}\right)^2}\right]$$

$$q' = -\left(\frac{R}{R+d}\right)Q \qquad q'' = -Q - q' = -Q + \frac{RQ}{R+d}$$
$$= \frac{-dQ}{R+d}$$

$$\therefore F = \frac{Q^2}{4\pi\varepsilon_0 (R+d)}\left[\frac{d}{(R+d)^2} + \frac{R}{\left(R+d-\frac{R^2}{R+d}\right)^2}\right]$$
$$= \frac{Q^2}{4\pi\varepsilon_0 (R+d)^3}\left[d + \frac{R}{\left[1 - R^2/(R+d)^2\right]^2}\right] \text{ ... enough. Yeuch.}$$

5. [15] A Leyden jar consists of a cylindrical beaker made of glass ($\varepsilon_r = 10$) of thickness $d = 0.2$ cm, with height $h = 20$ cm and radius $r = 10$ cm, coated on the inside and outside with a metal films, as indicated below. The top edges of the beaker are not coated. A charge $Q = +10^{-5}$ C is deposited on the inner metal using a van de Graaff generator with the outside of the beaker grounded. Estimate the voltage developed between the inner and outer metal films.



$$\text{Capacitance } C \simeq \frac{\varepsilon_r \varepsilon_0 A_{total}}{d} \quad \leftarrow \text{since } d \ll r, h$$
$$= \frac{\varepsilon_r \varepsilon_0 (\pi r^2 + 2\pi r h)}{d}$$

$$\therefore V \simeq \frac{Q}{C} \simeq \frac{Qd}{\pi \varepsilon_r \varepsilon_0 r(r+2h)}$$
$$\simeq \frac{10^{-5} \times 2\times10^{-3}}{3\times10\times10^{-11}\times10^{-1}\times(0.1+0.4)} \text{ V}$$
$$= \frac{2\times10^{-5-3-1+11+1}}{3\times0.5} \text{ V}$$
$$\simeq 1.3\times10^3 \text{ V} = 1.3 \text{ kV}$$

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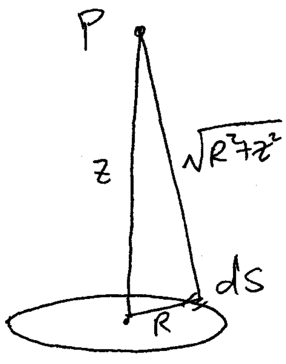
II. A ring of radius R centered on the origin in the x-y plane has uniform line charge density λ around its circumference. It is imbedded in an infinite, uniform dielectric of dielectric constant ε_r.

6. [20] Show that the potential along the z-axis is $V(z) = \dfrac{\lambda R}{2\varepsilon_r\varepsilon_0 z\sqrt{1+R^2/z^2}}$.



$\nabla^2 V = \dfrac{\rho_f}{\varepsilon_r \varepsilon_0}$ in uniform dielectric.

$\rightarrow V(\vec{r})$ is $\dfrac{1}{\varepsilon_r} \times$ (solution in vacuum)

ie $V(\vec{r}) = \displaystyle\int \frac{\rho(\vec{r}\,')(\vec{r}-\vec{r}\,')}{4\pi\varepsilon_r\varepsilon_0|\vec{r}-\vec{r}\,'|^2}\, d^3r'$

At point P, all elements ds of ring are distant by $|\vec{r}-\vec{r}\,'| = \sqrt{R^2+z^2}$

$$\therefore V_P = \oint_{\text{ring}} \frac{\lambda\, ds}{4\pi\varepsilon_r\varepsilon_0(z^2+R^2)^{1/2}} = \frac{2\pi R\lambda}{4\pi\varepsilon_r\varepsilon_0(z^2+R^2)^{1/2}}$$

$$= \frac{\lambda R}{2\varepsilon_r\varepsilon_0 z\left(1+R^2/z^2\right)^{1/2}}$$

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7. [20] By matching the result in the previous question to the eigenfunction expansion of the potential in spherical coordinates, find $V(r, \theta)$ away from the axis for $r > R$ to 3rd order in $1/r$.

$\nabla^2 V = 0$

In sph. coords, solution is

$$V = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

This must match $V(z) = \dfrac{\lambda R}{2\pi \varepsilon_r \varepsilon_0 z\left(1 + R^2/z^2\right)^{1/2}}$ along the z axis ie $\theta = 0$, $r = z$

$$\therefore \sum_{l=0}^{\infty}\left(A_l z^l + \frac{B_l}{z^{l+1}}\right) \underset{\text{unity}}{P_l(1)} = \frac{\lambda R}{2\pi \varepsilon_r \varepsilon_0 z\left(1 + R^2/z^2\right)^{1/2}}$$

binomial expansion for $R/z < 1$

$$\therefore \sum_{l=0}^{\infty}\left(A_l z^l + \frac{B_l}{z^{l+1}}\right) = \frac{\lambda R}{2\pi \varepsilon_r \varepsilon_0} \cdot \frac{1}{z}\left[1 - \frac{1}{2}\frac{R^2}{z^2} + O\left(\frac{R}{z}\right)^4\right]$$

Equate coeffs of z^l: $A_l = 0$ for all l

$B_0 = \dfrac{\lambda R}{2\pi \varepsilon_r \varepsilon_0}$ $\quad B_1 = 0 \quad B_2 = -\dfrac{R^2}{2} B_0 \ldots$

$$\therefore V = \frac{\lambda R}{2\pi \varepsilon_r \varepsilon_0}\left[\frac{1}{r} P_0(0) - \frac{R^2}{2r^3} P_2(\cos\theta) \ldots\right] = \frac{\lambda R}{2\pi \varepsilon_r \varepsilon_0}\left[\frac{1}{r} - \frac{R^2}{2r^3}\left(\frac{3\cos^2\theta - 1}{2}\right) + \ldots\right]$$

8. [10] Give the definitions of the monopole moment Q and the dipole moment $\mathbf{p}$ for some localized charge distribution $\rho(\mathbf{r})$. When is $\mathbf{p}$ independent of coordinate system?

$$Q = \int \rho(\vec{r})\, d^3r \qquad \vec{p} = \int \vec{r} \rho(\vec{r})\, d^3r$$

$\vec{p}$ is indep of origin when $Q = 0$.

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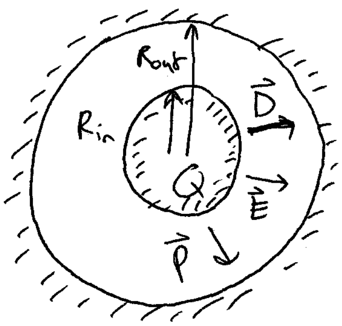
III. A spherical capacitor consists of two concentric spherical metal shells of radii R_{in} and R_{out} with the space between them filled by a uniform linear dielectric of dielectric constant ε_r. The inner shell holds a charge Q and the outer is grounded.

9. [25] Find **E**, **D** and **P** within the dielectric, all bound charge, and the capacitance C.



By symmetry $\vec{E} = E(r)\hat{r}$

$\vec{D} = D(r)\hat{r}$

$\vec{P} = P(r)\hat{r}$

$\vec{\nabla}\cdot\vec{D} = \rho_f$ & Gauss's law over sphere radius r

$\rightarrow 4\pi r^2 D = Q \quad \therefore \; D(r) = \dfrac{Q}{4\pi r^2}$

$\vec{D} = \varepsilon_r \varepsilon_0 \vec{E} \longrightarrow E(r) = \dfrac{Q}{4\pi\varepsilon_r\varepsilon_0 r^2}$

$\vec{P} = (\varepsilon_r - 1)\varepsilon_0\vec{E} \longrightarrow P(r) = \dfrac{(\varepsilon_r - 1)Q}{4\pi\varepsilon_r r^2}$

$\rho_b = \vec{\nabla}\cdot\vec{P} = 0$ in the bulk dielectric

At $r = R_{in}$, $\vec{P}\cdot\hat{n} = -P(R_{in}) = \dfrac{(1-\varepsilon_r)Q}{4\pi\varepsilon_r R_{in}^2}$ surface bound σ_b charge at inner conductor

At $r = R_{out}$, $\vec{P}\cdot\hat{n} = +P(R_{out}) = \dfrac{(\varepsilon_r - 1)Q}{4\pi\varepsilon_r R_{out}^2}$ " " at outer "

$$V = \int_{R_{out}}^{R_{in}} -E(r)\,dr = \frac{Q}{4\pi\varepsilon_r\varepsilon_0}\int_{R_{out}}^{R_{in}} \frac{-dr}{r^2} = \frac{Q}{4\pi\varepsilon_r\varepsilon_0}\left(\frac{1}{R_{in}} - \frac{1}{R_{out}}\right)$$

$$\therefore C = \frac{Q}{V} = 4\pi\varepsilon_r\varepsilon_0\left(\frac{1}{R_{in}} - \frac{1}{R_{out}}\right)^{-1}$$

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10. [12] Show that the energy U stored in the dielectric is equal to $Q^2/2C$.

$$U = \int \tfrac{1}{2}\vec{D}\cdot\vec{E}\, d^3r = \int_{R_{in}}^{R_{out}} \frac{1}{2}\left(\frac{Q}{4\pi r^2}\right)\left(\frac{Q}{4\pi\varepsilon_r\varepsilon_0 r^2}\right)\cdot 4\pi r^2 dr$$

$$= \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_r\varepsilon_0}\int_{R_{in}}^{R_{out}}\frac{dr}{r^2} = \frac{Q^2}{2}\cdot\frac{1}{4\pi\varepsilon_r\varepsilon_0}\left(\frac{1}{R_{in}} - \frac{1}{R_{out}}\right)$$

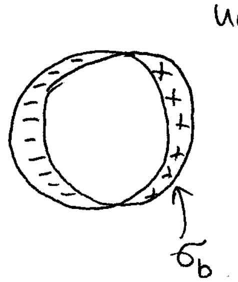
$$= \frac{Q^2}{2C}.$$

11. [5] What is the electrostatic pressure on the outer shell?

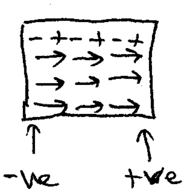
$$P = \frac{\sigma_f^2}{2\varepsilon_0} = \frac{D(R_{out})^2}{2\varepsilon_0} = \frac{1}{2\varepsilon_0}\left(\frac{Q}{4\pi R_{out}^2}\right)^2 = \frac{Q^2}{32\pi^2\varepsilon_0 R_{out}^4}$$

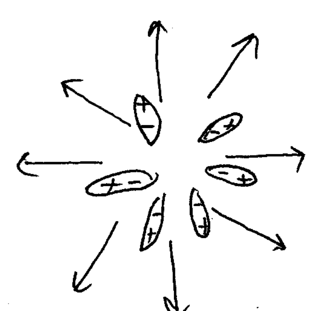
12. [8] Draw cartoons to indicate why $\mathbf{P}.\hat{\mathbf{n}} = \sigma_b$ at the surface of a dielectric and $\nabla.\mathbf{P} = -\rho_b$ in the bulk.

uniform $\vec{P}$ = uniform -ve displaced w.r.t. uniform positive → imbalance at surfaces $\vec{P}.\hat{n}$



or





$\vec{\nabla}.\vec{P} > 0 \Rightarrow$ dipoles point outwards → net -ve charge density

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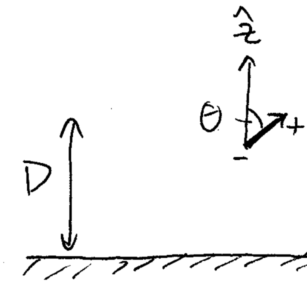
IV. A point dipole **p** is located a distance D above the conducting surface $z = 0$, oriented at an angle θ to the z-axis.

13. [12] Find an expression for the total potential $V(\mathbf{r})$ by using the method of images (think of the point dipole as a physical dipole).



$\vec{p} = (p_x, p_y, p_z)$ real dipole at $\vec{r}_1 = (0,0,D)$

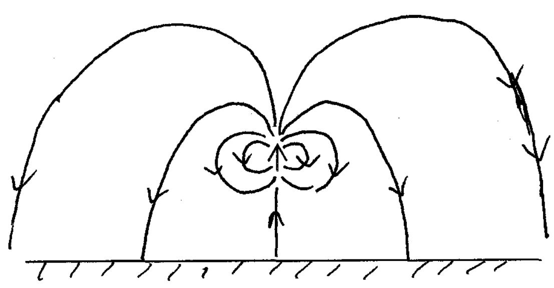
$$V(\vec{r}) = V_{real} + V_{image}$$

$$= \frac{\vec{p}\cdot(\vec{r}-\vec{r}_1)}{4\pi\varepsilon_0|\vec{r}-\vec{r}_1|^3} + \frac{\vec{p}_{im}\cdot(\vec{r}+\vec{r}_1)}{4\pi\varepsilon_0|\vec{r}+\vec{r}_1|^3}$$

$\vec{p}_{im} = (-p_x, -p_y, p_z)$ at $-\vec{r}_1 = (0,0,-D)$

14. [12] Sketch the field lines

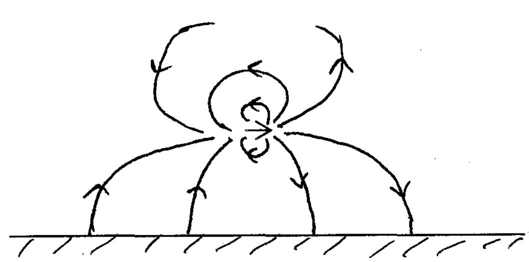
(i) for $\theta = 0$



close up - dipole $\vec{p}$
far away - dipole $2\vec{p}$
$\vec{E}$ must be normal at metal surface.

$\uparrow \vec{p}_{im}$

(ii) for $\theta = \pi/2$.



close up - horizontal dipole $\vec{p}$.

far away - quadrupole

$\rightarrow \vec{p}$
$\leftarrow -\vec{p}$

$\leftarrow \vec{p}_{im}$

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15. [12] Find an expression for the net force on the dipole valid for any θ.

$$\vec{F} = -\vec{\nabla}_{r_1}(\vec{p}\cdot\vec{E}) = -\vec{\nabla}_{r_1}\left\{\frac{\vec{p}\cdot\left[(\vec{p}_{im}\cdot\hat{r}_1)\hat{r}_1 - \vec{p}_{im}\right]}{4\pi\varepsilon_0|2\vec{r}_1|^2}\right\}_{\vec{r}_1=(0,0,D)}$$

$$= -\vec{\nabla}_{r_1}\left\{\frac{\vec{p}\cdot\left[(\vec{p}_{im}\cdot\hat{z})\hat{z} - \vec{p}_{im}\right]}{16\pi\varepsilon_0 r_1^2}\right\}$$

would someone please check this?!

$$= -\vec{\nabla}_{r_1}\frac{\left(p_z p_z - \vec{p}\cdot\vec{p}_{im}\right)}{16\pi\varepsilon_0 r_1^2} \qquad \vec{p}_{im}\cdot\hat{z} = p_z = \vec{p}\cdot\hat{z}$$

$$= \frac{\hat{r}_1}{8\pi\varepsilon_0 r_1^3}\left(p^2\cos^2\theta + p^2\cos 2\theta\right) = \frac{\hat{z}\,p^2\left(\cos^2\theta + \cos 2\theta\right)}{8\pi\varepsilon_0 D^3}$$

16. [14] Find an expression for the torque valid for any θ. Show that the torque is zero at both $\theta = 0$ and $\theta = \pi/2$.

$$\vec{N} = \vec{p}\times\underbrace{\vec{E}(\vec{r}_1)}_{\text{due to image}} = \vec{p}\times -\vec{\nabla}_r\left[\frac{\vec{p}_{im}\cdot(\vec{r}+\vec{r}_1)}{4\pi\varepsilon_0|\vec{r}+\vec{r}_1|^3}\right]_{\vec{r}=\vec{r}_1}$$

$$= \vec{p}\times\left[\frac{(\vec{p}_{im}\cdot\hat{z})\hat{z} - \vec{p}_{im}}{4\pi\varepsilon_0(2D)^2}\right]$$

For $\theta = 0$, $\vec{p}$ and $\vec{p}_{im}$ and $\vec{r}_1$ are all parallel so cross products vanish $\rightarrow \vec{N} = 0$

For $\theta = \frac{\pi}{2}$, $\vec{p}\times\hat{z} = 0$ and $\vec{p}\times\vec{p}_{im} = \vec{p}\times(-\vec{p}) = 0$ so $\vec{N} = 0$.