 Name solutions

Electrodynamics, Physics 321 **Second midterm** **11.30 am Wed 1 March 06**
Winter 2006 Instructor: David Cobden

Do not turn this page until I say 'go' at 11.30. You have 50 minutes. Hand your exam to me before I leave the room at 12.25.

This exam contains 100 points. Be sure to attempt all the questions.

Please write your name on every page and your SID on the first page.

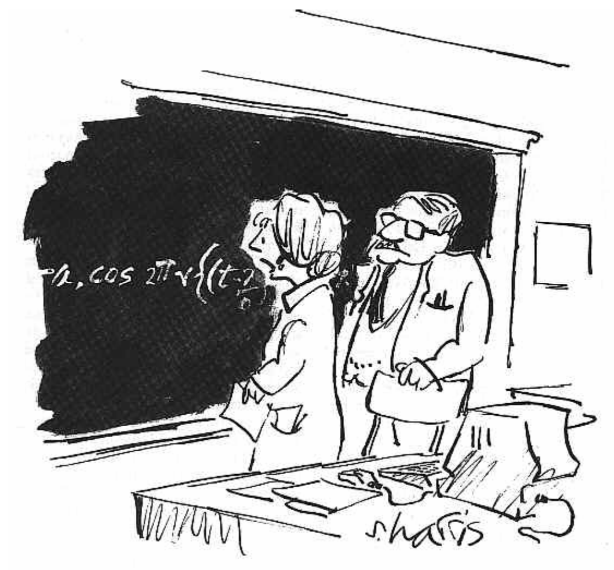
Write all your working on these question sheets. Use this front page for extra working. It is important to show your calculation or derivation. You probably won't get full marks just for stating the correct answer if you don't show how you get it.

Watch the blackboard/overhead for corrections or clarifications during the exam.

This is a closed book exam. No books, notes or calculators allowed.

$\int_{-1}^{1} P_l(x)P_m(x)dx = \frac{2}{2l+1}\delta_{mn}$ $\quad P_0(x) = 1 \quad P_1(x) = x \quad P_2(x) = (3x^2-1)/2 \quad P_3(x) = (5x^3 - 3x)/2$

$$\nabla\left(\frac{\mathbf{p}.\hat{\mathbf{r}}}{r^2}\right) = \frac{3(\mathbf{p}.\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{r^3}$$



"IT'S AN EXCELLENT PROOF, BUT IT LACKS WARMTH AND FEELING."

 Name solutions

1. [10] State the uniqueness theorem for Laplace's equation as carefully as you can (the general one we proved in class, not one of the special ones in Griffiths). Specify the types of boundary conditions that are sufficient (you don't need to remember their names).

If either V or $\frac{\partial V}{\partial n}$ is known at every point on a surface S bounding a region $\mathcal{V}$, then $V(\vec{r})$ is uniquely determined everywhere in $\mathcal{V}$. So, if you find a solution $V(\vec{r})$ that matches the b.c.'s it must be the correct one.

2. [10] A conducting sphere of radius R carries a positive charge Q. A particle with positive charge q is placed somewhere near the sphere. Show that the particle is repelled from the sphere when it is far away and attracted when it is close.

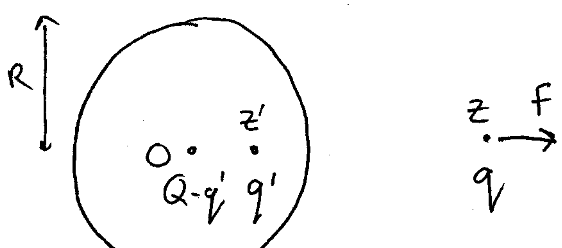


Image charges are $Q-q'$ at origin and q' at z'

where $q' = -\frac{R}{z}q$ and $z' = \frac{R^2}{z}$

$$\therefore F = \frac{q(Q-q')}{4\pi\varepsilon_0 z^2} + \frac{qq'}{4\pi\varepsilon_0 (z-z')^2}$$

For $z \gg R$, $|q'| \ll q$ and $z - z' \simeq z$ $\therefore$ first term dominates

$$F \simeq \frac{qQ}{4\pi\varepsilon_0 z^2}$$ (effect of polarization of sph. is negligible)

For $z - R \ll R$, $q' \simeq -q$ and $z \simeq R$ and $z - z' \ll R$ $\therefore$ 2nd term dominates

$$F \simeq \frac{-q^2}{4\pi\varepsilon_0 (z - R^2/z)^2}$$ (attraction to image > repulsion by Q)

3. [10] Find an implicit equation for the distance z_c (between particle and center of sphere) at which the force between particle and sphere vanishes. Simplify the equation but do not try to solve for z_c.

At $z = z_c$, $F = 0$ $\rightarrow$ $\frac{Q-q'}{z_c^2} = \frac{-q'}{(z_c - R^2/z_c)^2}$ $\therefore$ $\frac{Q + \frac{Rq}{z_c}}{z_c^2} = \frac{Rq/z_c}{z_c^2(1 - R^2/z_c^2)^2}$

$$\therefore \left(Q + \frac{Rq}{z_c}\right)\left(1 - \frac{R^2}{z_c^2}\right)^2 = \frac{Rq}{z_c}$$

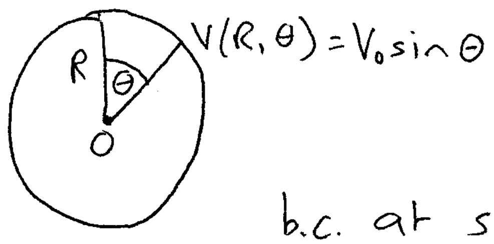
or $\left(\frac{Q}{q} + \alpha\right)(1-\alpha^2)^2 = \alpha$ where $\alpha = \frac{R}{z_c}$

Name solutions

4. [10] A spherical cavity of radius R has an azimuthally symmetric potential $V_0 \sin\theta$ maintained over its inner surface, where θ is the usual polar angle. Write down the appropriate general eigenfunction expansion for $V(r,\theta)$ in the cavity.

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

5. [20] Identify all coefficients in the expansion which are zero (don't forget to account for symmetry of the Legendre polynomials), and evaluate the first nonzero coefficient.



V finite as $r \to 0 \Rightarrow B_l = 0$ for all $l \geq 0$

$$\therefore V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

b.c. at surface: $V_0 \sin\theta = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta)$

Fourier trick: $\int_{-1}^{1} V_0 \sin\theta \, P_m(\cos\theta)\, d(\cos\theta) = A_m R^m \cdot \frac{2}{2m+1}$

$$\therefore A_l = \frac{2l+1}{2} \frac{V_0}{R} \int_0^{\pi} P_l(\cos\theta) \sin^2\theta \, d\theta$$

For odd l, P_l is odd about $\theta = \frac{\pi}{2}$ but $\sin^2\theta$ is even

$\therefore A_l = 0$ for l odd.

1st coeff: for $l = 0$, $A_0 = \frac{1}{2} \frac{V_0}{R^0} \int_0^{\pi} \sin^2\theta \, d\theta = \frac{\pi}{4} V_0$

(use av. of $\sin^2\theta = \frac{1}{2}$)

6. [10] Now consider the same cavity, with the same potential on its surface, but with a point charge q located at its center. What is the difference in the eigenfunction expansion for this case compared with the situation with no point charge?

Now $V \to \frac{q}{4\pi\varepsilon_0 r}$ as $r \to 0$ $\therefore$ must have $B_0 = \frac{q}{4\pi\varepsilon_0}$

while $B_{l>0} = 0$ still.

To keep $V = V_0 \sin\theta$ at $r = R$ we also need $A_0 = \frac{-q}{4\pi\varepsilon_0 R}$

while $A_{l>0}$ is as above.

 Name solutions

7. [10] Show that the dipole moment of a an object with a rigid charge distribution is independent of choice of origin only if the total charge is zero.

$\vec{p} = \sum_k q_k \vec{r}_k$ If we shift origin by $\vec{a}$, (do it in integral form if you like)

$$\vec{p}' = \sum_k q_k(\vec{r}_k - \vec{a}) = \sum_k q_k \vec{r}_k - a \underbrace{\sum_k q_k}_{\text{total charge } Q}$$

$= \vec{p}$ if $Q = 0$

8. [10] A small simple dipole **p** is located a distance *d* from and parallel to the flat surface of a conductor, as indicated below. Using the image charge method (sketch as required on the diagram below), find an expression for the electric potential $V(\mathbf{r})$ outside the conductor (not valid very close to the dipole).

Uniqueness → image dipole $-\vec{p}$ at $\overbrace{(0,0,-d)}^{=-\vec{d}}$ does the job.

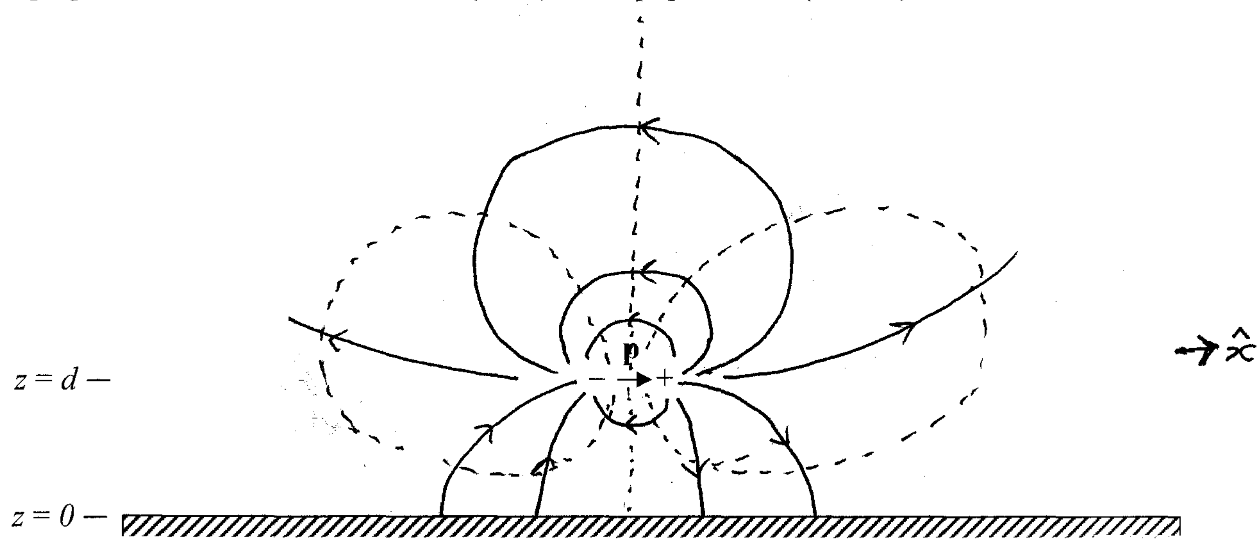
$\vec{p}$ is at $(0,0,d) = \vec{d}$

$\vec{p} = (p, 0, 0)$

$$\therefore V(\vec{r}) = \frac{\vec{p}\cdot(\vec{r}-\vec{d})}{4\pi\varepsilon_0 (\vec{r}-\vec{d})^3} + \frac{(-\vec{p})\cdot(\vec{r}+\vec{d})}{4\pi\varepsilon_0(\vec{r}+\vec{d})^3}$$

$$= \frac{px}{4\pi\varepsilon_0[x^2+y^2+(z-d)^2]^{3/2}} - \frac{px}{4\pi\varepsilon_0[x^2+y^2+(z+d)^2]^{3/2}}$$

9. [10] Sketch the electric field lines (solid) and equipotentials (dashed).



$+ \leftarrow -$
$(-\vec{p})$ image dipole

10. [5 extra on top of 100] Find an expression for the charge density on the surface of the conductor.

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial z} = -\frac{3px}{8\pi}\left[\frac{(2z-2d)}{[x^2+y^2+(z-d)^2]^{5/2}} - \frac{2(z+d)}{[x^2+y^2+(z+d)^2]^{5/2}}\right]$$