Equations of Physics 321 - Electrostatics

You should know these equations, the relations between them, and their derivations.

$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$ and $\nabla \times \mathbf{a}(\mathbf{r})$ and $\nabla^2 f(\mathbf{r})$ in Cartesian coordinates only

$dxdydz = r^2 \sin\theta\, dr\, d\theta\, \delta\phi = s\, ds\, d\phi\, dz \quad (x,y,z) = r(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\phi) = (s\cos\phi, s\sin\phi, z)$

Other derivatives and identities will be given in the exam if needed.

Point charge q at $\mathbf{r}_k$: $\rho(\mathbf{r}) = q\delta^3(\mathbf{r}-\mathbf{r}_k) \quad \delta^3(\mathbf{r}) = \nabla.\left(\frac{\hat{\mathbf{r}}}{4\pi r^2}\right) \quad V(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}_k|} \quad \mathbf{E}(\mathbf{r}) = \frac{q(\mathbf{r}-\mathbf{r}_k)}{4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}_k|^3}$

Charge distribution: $\mathbf{E}(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}'|^3} d^3r' \quad \nabla.\mathbf{E} = \frac{\rho}{\varepsilon_0} \quad \mathbf{E} = -\nabla V \quad V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}.d\mathbf{l}$

Gauss: $\oint_S \mathbf{E}.d\mathbf{S} = \int_{\text{inside S}} \nabla.\mathbf{E} d^3r = \int \frac{\rho}{\varepsilon_0} d^3r = \frac{Q_{enc}}{\varepsilon_0}$ Apply this to symmetric charge densities.

Stokes: $\oint_{\text{any}} \mathbf{E}.d\mathbf{l} = \int_S \nabla\times\mathbf{E}.d\mathbf{S} = 0 \quad \nabla\times\mathbf{E} = 0$ (ie, **E** is conservative or irrotational)

Poisson's equation: $\nabla^2 V = -\frac{\rho}{\varepsilon_0} \quad G_{\nabla^2} = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} \quad V = \int G_{\nabla^2}(\mathbf{r}-\mathbf{r}')\frac{\rho(\mathbf{r}')}{\varepsilon_0} d^3r' = \int \frac{\rho(\mathbf{r}')}{4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}'|} d^3r'$

Electrostatic energy: $U = \frac{1}{2}\int_{\text{all space}} \rho(\mathbf{r})V(\mathbf{r})d^3r = \frac{1}{2}\int \varepsilon_0 E^2 d^3r = \int_{\text{all space}} u_E d^3r \quad u_E = \frac{\varepsilon_0 E^2}{2}$

Boundary conditions at an interface: $\mathbf{E}_2 - \mathbf{E}_1 = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}$ Parallel plate capacitor: $C = \frac{\varepsilon_0 A}{d}$

Metals: **E** = 0 inside. $\mathbf{E}_{surface} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}$ Electrostatic pressure: $\frac{\mathbf{F}}{A} = \frac{\sigma^2}{2\varepsilon_0}\hat{\mathbf{n}}$ Capacitance: $C = \frac{Q}{V} \quad U = \frac{CV^2}{2}$

State and understand the uniqueness theorem for Dirichlet and Neumann boundary conditions.

Image of point charge q at $\mathbf{r}_1$ in the conducting plane $z = 0$ is $-q$ at $-\mathbf{r}_1$.

Image of point charge q at $\mathbf{r}_1$ in the neutral conducting sphere $r=R$ is $q' = -qR/r_1$ at $\mathbf{r}_2 = (R/r_1)^2\mathbf{r}_1$ and $-q'$ at $\mathbf{r} = 0$.

$\nabla^2 V = 0 \Rightarrow$ in spherical coords with azimuthal symmetry, $V(r,\theta) = \sum_{l=0}^{\infty}(A_l r^l + \frac{B_l}{r^{l+1}})P_l(\cos\theta)$

in Cartesian coords: $V(x,y,z) = \sum_{k_1,k_2}^{\infty}(A_i e^{k_1 x} + B_i e^{-k_1 x})(C_i e^{k_2 y} + D_i e^{-k_2 y})(E_i e^{k_3 z} + F_i e^{-k_3 z}) \quad ; \; k_1^2 + k_2^2 + k_3^2 = 0$

$\int_{-a}^{a} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)dx = a\delta_{mn} \quad \int_{-1}^{1} P_l(x)P_m(x)dx = \frac{2}{2l+1}\delta_{mn} \quad P_0(x) = 1 \quad P_1(x) = x \quad P_l(x) = (-1)^l P_l(-x)$

Dipole moment: $\mathbf{p} = \int \mathbf{r}\rho(\mathbf{r})d^3r$ potential of dipole: $V_{dip}(\mathbf{r}) = \frac{\mathbf{p}.\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2} = \frac{qd\cos\theta}{4\pi\varepsilon_0 r^2}$ for (-q)—d—(q)

For fixed dipole in E-field, Energy: U = **-p.E** Force: $\mathbf{F} = -\nabla U = \nabla(\mathbf{p}.\mathbf{E})$ Torque: **N** = **p** × **E**

Polarizability of molecule: $\mathbf{p} = \alpha\mathbf{E}$ $\alpha = 4\pi\varepsilon_0\times$(characteristic volume) $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$ $\nabla.\mathbf{P} = -\rho_b$ $\mathbf{P}.\hat{\mathbf{n}} = \sigma_b$

Linear dielectric: $\mathbf{P} = \chi\varepsilon_0\mathbf{E}$ $\mathbf{D} = (1+\chi)\varepsilon_0\mathbf{E} = \varepsilon_r\varepsilon_0\mathbf{E}$ $\nabla^2 V = \frac{\rho_f}{\varepsilon_r\varepsilon_0}$ $\rho_b = \rho - \rho_f = \left(\frac{1}{\varepsilon_r} - 1\right)\rho_f$ Energy density: $\frac{\mathbf{D}.\mathbf{E}}{2}$

Boundary conditions between dielectrics: $E_1^{\parallel} = E_2^{\parallel} \rightarrow V_1 = V_2$ and $D_1^{\perp} - D_2^{\perp} = \sigma_b \rightarrow \varepsilon_{r1}\frac{\partial V_1}{\partial n} = \varepsilon_{r2}\frac{\partial V_2}{\partial n}$ if $\sigma_b = 0$.

Conductor corresponds to $\varepsilon_r \rightarrow \infty$ Parallel plate capacitor: $C = \varepsilon_r\varepsilon_0 A/d$

Force to move piece of dielectric out of capacitor = $F_x = \frac{dU_{total}}{dx} = -\frac{V^2}{2}\frac{dC}{dx} \quad dU_{total} = dU_{battery} + d\left(\frac{QV}{2}\right)\Big|_{Q \text{ or } V}$