321 Homework 9 solutions

P 4.19

From Eq. 2.54, the capacitance with no dielectric is $C_0 = A\epsilon_0/d$, where A is the area of the plates and d is the separation between the plates.

In configuration (a), suppose the charge density is σ on the upper plate, $-\sigma$ on the lower, then $D = \sigma$ between the plates (this is completely analogous to Ex 2.5, but with $\underline{D}$ determined from Eq. 4.23)

Now $E = \sigma/\epsilon_0$ in air, and $E = \sigma/\epsilon$ in dielectric, so the potential difference can be expressed as $V = \frac{\sigma}{\epsilon_0}\frac{d}{2} + \frac{\sigma}{\epsilon}\frac{d}{2} = \frac{Q}{A}\frac{d}{2}\frac{1}{\epsilon_0}\left(1 + \frac{1}{\epsilon/\epsilon_0}\right) = \frac{Qd}{2A\epsilon_0}(1+\epsilon_0/\epsilon) = \frac{\sigma}{\epsilon_0}\frac{1+\epsilon_r^{-1}}{2}$

Hence $C_a = \frac{Q}{V} = \frac{2A\epsilon_0}{d}\frac{1}{1+\epsilon_r^{-1}} = \frac{2A\epsilon_0}{d}\frac{\epsilon_r}{\epsilon_r+1}$ $\therefore \frac{C_a}{C_0} = \frac{2\epsilon_r}{1+\epsilon_r}$

In configuration (b), given that the potential difference is V, $E = V/d$, so $\sigma = \epsilon_0 E = \epsilon_0 V/d$ for the part with no dielectric.

Next $P = \epsilon_0\chi_e E = \epsilon_0\chi_e V/d$ in dielectric, so $\sigma_b = \underline{P}\cdot\hat{\underline{n}} = -\epsilon_0\chi_e V/d$ at top surface of dielectric ($\hat{n}$ points "up" while $\underline{E}$ points "down").

Since V is the same, $\sigma = \sigma_{tot} = \sigma_b + \sigma_f$ for the dielectric part. Hence $\sigma_f = \sigma - \sigma_b = \epsilon_0 V\underbrace{(1+\chi_e)}_{\epsilon_r}/d$ on the top plate above the dielectric.

Thus $C_b = \frac{Q}{V} = \frac{1}{V}\left(\sigma\frac{A}{2} + \sigma_f\frac{A}{2}\right) = \frac{A}{2V}\left(\epsilon_0\frac{V}{d} + \epsilon_0\epsilon_r\frac{V}{d}\right) = \frac{A\epsilon_0}{2d}(1+\epsilon_r)$ $\therefore \frac{C_b}{C_0} = \frac{1+\epsilon_r}{2}$

Let z axis points "down"

		$\underline{E}$	$\underline{D}$	$\underline{P}$	σ_b (top surface)	σ_f (top plate)
Config (a)	Air	$\frac{2\epsilon_r}{1+\epsilon_r}\frac{V}{d}\hat{\underline{z}}$	$\frac{2\epsilon_r}{1+\epsilon_r}\frac{\epsilon_0 V}{d}\hat{\underline{z}}$	0	0	$\frac{2\epsilon_r}{1+\epsilon_r}\frac{\epsilon_0 V}{d}$
	Dielectric	$\frac{2}{1+\epsilon_r}\frac{V}{d}\hat{\underline{z}}$	$\frac{2\epsilon_r}{1+\epsilon_r}\frac{\epsilon_0 V}{d}\hat{\underline{z}}$	$2\frac{\epsilon_r-1}{\epsilon_r+1}\frac{\epsilon_0 V}{d}\hat{\underline{z}}$	$-2\frac{\epsilon_r-1}{\epsilon_r+1}\frac{\epsilon_0 V}{d}$	N.A.
Config (b)	Air	$\frac{V}{d}\hat{\underline{z}}$	$\frac{\epsilon_0 V}{d}\hat{\underline{z}}$	0	0	$\frac{\epsilon_0 V}{d}$ (left)
	Dielectric	$\frac{V}{d}\hat{\underline{z}}$	$\epsilon_r\frac{\epsilon_0 V}{d}\hat{\underline{z}}$	$(\epsilon_r-1)\frac{\epsilon_0 V}{d}\hat{\underline{z}}$	$-(\epsilon_r-1)\frac{\epsilon_0 V}{d}$	$\epsilon_r\frac{\epsilon_0 V}{d}$ (right)

P4.24

Analogous to Ex 4.7. Here we have

$$\text{B.C.'s}: \begin{cases} (1) & V_{out} = V_{med} & \text{at } r=b \\ (2) & \epsilon\,\partial_r V_{med} = \epsilon_0 \partial_r V_{out} & \text{at } r=b \\ (3) & V_{med} = 0 & \text{at } r=a \end{cases}$$

$$V(r,\theta) = \begin{cases} V_{out}(r,\theta) = -E_0 r\cos\theta + \sum_\ell B_\ell r^{-(\ell+1)} P_\ell(\cos\theta) & , r>b \\ V_{med}(r,\theta) = \sum_\ell \left(A_\ell r^\ell + \tilde{B}_\ell r^{-(\ell+1)}\right) P_\ell(\cos\theta) & , a<r<b \\ V_{in}(r,\theta) = 0 & , r<a \end{cases}$$

$$(1) \Rightarrow -E_0 b\cos\theta + \sum_\ell B_\ell b^{-(\ell+1)} P_\ell(\cos\theta) = \sum_\ell \left(A_\ell b^\ell + \tilde{B}_\ell b^{-(\ell+1)}\right) P_\ell(\cos\theta)$$

$$(2) \Rightarrow \epsilon_r \sum_\ell \left[\ell A_\ell b^{\ell-1} - (\ell+1)\tilde{B}_\ell b^{-(\ell+2)}\right] P_\ell(\cos\theta) = -E_0\cos\theta - \sum_\ell (\ell+1) B_\ell b^{-(\ell+2)} P_\ell(\cos\theta)$$

$$(3) \Rightarrow A_\ell a^\ell + \tilde{B}_\ell a^{-(\ell+1)} = 0 \Rightarrow \tilde{B}_\ell = -a^{(2\ell+1)} A_\ell$$

For $\ell \neq 1$:

$$\left.\begin{aligned} &(1): B_\ell b^{-(\ell+1)} = A_\ell\left(b^\ell - \frac{a^{2\ell+1}}{b^{\ell+1}}\right) \Rightarrow B_\ell = A_\ell\left(b^{2\ell+1} - a^{2\ell+1}\right) \\ &(2): \epsilon_r\left[\ell A_\ell b^{\ell-1} + (\ell+1)\frac{a^{2\ell+1}}{b^{\ell+2}} A_\ell\right] = -(\ell+1) B_\ell b^{-(\ell+2)} \Rightarrow B_\ell = -\epsilon_r A_\ell\left[\frac{\ell}{\ell+1} b^{2\ell+1} + a^{2\ell+1}\right] \end{aligned}\right\} \Rightarrow A_\ell = B_\ell = 0$$

For $\ell = 1$:

$$\left.\begin{aligned} &(1): -E_0 b + B_1 b^{-2} = A_1\left(b - \frac{a^3}{b^2}\right) \Rightarrow B_1 - E_0 b^3 = A_1 (b^3 - a^3) \\ &(2): \epsilon_r\left(A_1 + 2\frac{a^3 A_1}{b^3}\right) = -E_0 - 2\frac{B_1}{b^3} \Rightarrow -2B_1 - E_0 b^3 = \epsilon_r A_1 (b^3 + 2a^3) \end{aligned}\right\} \Rightarrow A_1 = \frac{-3E_0}{2[1-(a/b)^3] + \epsilon_r[1+2(a/b)^3]}$$

$$\therefore \mathbf{E}_{med}(r,\theta) = -\nabla V_{med} = \frac{3E_0}{2[1-(a/b)^3] + \epsilon_r[1+2(a/b)^3]}\left\{\left(1+\frac{2a^3}{r^3}\right)\cos\theta\,\hat{r} - \left(1-\frac{a^3}{r^3}\right)\sin\theta\,\hat{\theta}\right\}$$

P4.26

From Ex 4.25:

$$\mathbf{D} = \begin{cases} 0 & r<a \\ \dfrac{Q}{4\pi r^2}\hat{r} & r>a \end{cases}, \quad \mathbf{E} = \begin{cases} 0 & r<a \\ (Q/4\pi\epsilon r^2)\,\hat{r} & a<r<b \\ (Q/4\pi\epsilon_0 r^2)\,\hat{r} & r>b \end{cases}$$

$$W = \frac{1}{2}\int d\tau\, \mathbf{D}\cdot\mathbf{E} = \frac{1}{2}\left(\frac{Q}{4\pi}\right)^2 4\pi\left\{\frac{1}{\epsilon}\int_a^b dr\, r^2\frac{1}{r^4} + \frac{1}{\epsilon_0}\int_b^\infty dr\, r^2\frac{1}{r^4}\right\} = \frac{Q^2}{8\pi}\left\{\epsilon^{-1}(r^{-1})\Big|_a^b + \epsilon_0^{-1}(r^{-1})\Big|_b^\infty\right\}$$

$$= \frac{Q^2}{8\pi\epsilon_0}\left\{\frac{1}{1+\chi_e}(a^{-1}-b^{-1}) + b^{-1}\right\} = \frac{Q^2}{8\pi\epsilon_0(1+\chi_e)}\left(\frac{1}{a} + \frac{\chi_e}{b}\right)$$

P4.28

From Eq 4.64, the upward force on the oil is $F = (V^2/2)(dC/dh)$, so first find the capacitance.

Air part: $E = 2\lambda/4\pi\epsilon_0 s \Rightarrow V = \frac{2\lambda}{4\pi\epsilon_0}\ln(b/a)$, λ charge density of inner tube in air part

Oil part: $D = 2\lambda/4\pi s \Rightarrow E = 2\lambda'/4\pi\epsilon s \Rightarrow V = \frac{2\lambda'}{4\pi\epsilon}\ln(b/a)$

Since V is the same, $\lambda/\epsilon_0 = \lambda'/\epsilon \Rightarrow \lambda' = (\epsilon/\epsilon_0)\lambda = \epsilon_r\lambda$.

Now $Q = \lambda' h + \lambda(l-h) = \epsilon_r\lambda h - \lambda h + \lambda l = \lambda[(\epsilon_r - 1)h + l] = \lambda(\chi_e h + l)$, l total height.

Hence $C = \frac{Q}{V} = \frac{\lambda(\chi_e h + l)}{2\lambda \ln(b/a)} 4\pi\epsilon_0 = 2\pi\epsilon_0 \frac{\chi_e h + l}{\ln(b/a)}$ $\therefore F = \frac{1}{2}V^2 \frac{2\pi\epsilon_0\chi_e}{\ln(b/a)}$

The oil will rise until the upward force is balanced by that from gravity $F = mg = \rho\pi(b^2-a^2)gh$

$\therefore h = \frac{\epsilon_0\chi_e V^2}{\rho(b^2-a^2)g\ln(b/a)}$

P4.32

$\oint d\mathbf{a}\cdot\mathbf{D} = Q_{f,enc.} \Rightarrow \mathbf{D} = \frac{q}{4\pi r^2}\hat{r}$, $\mathbf{E} = \epsilon^{-1}\mathbf{D} = \frac{q}{4\pi\epsilon_0(1+\chi_e)}\frac{\hat{r}}{r^2}$, $\mathbf{P} = \epsilon_0\chi_e\mathbf{E} = \frac{q\chi_e}{4\pi(1+\chi_e)}\frac{\hat{r}}{r^2}$

$\rho_b = -\nabla\cdot\mathbf{P} = -\frac{q\chi_e}{4\pi(1+\chi_e)}\left(\nabla\cdot\frac{\hat{r}}{r^2}\right) = -q\frac{\chi_e}{1+\chi_e}\delta^3(\mathbf{r})$ by Eq 1.99, $\sigma_b = \mathbf{P}\cdot\hat{r} = \frac{q\chi_e}{4\pi(1+\chi_e)}\frac{1}{R^2}$

$Q_{surf} = \sigma_b(4\pi R^2) = q\frac{\chi_e}{1+\chi_e}$. The compensating -ve charge is at the center $\int d\tau\rho_b = -\frac{q\chi_e}{1+\chi_e}\underbrace{\int d\tau\,\delta^3(\mathbf{r})}_{=1}$

P4.36

(a) Propose $V(r) = V_0 R/r$, then $\mathbf{E} = -\nabla V = (V_0 R/r^2)\hat{r}$, $\mathbf{P} = \epsilon_0\chi_e V_0\frac{R}{r^2}\hat{r}$ for $z<0$ ($\mathbf{P}=0$ for $z>0$),

and $\sigma_b = \epsilon_0\chi_e V_0\frac{R}{R^2}(\hat{r}\cdot\hat{n}) = -\frac{\epsilon_0\chi_e V_0}{R}$ ($\hat{n}$ points out of the dielectric $\Rightarrow \hat{n} = -\hat{r}$). This σ_b is on

the surface at $r=R$. The flat surface $z=0$ carries no bound charge since $\hat{n} = \hat{z} \perp \hat{r}$ here.
There is also no bound volume charge (by Eq 4.39, since $\rho_f = 0$). If V is to have the required spherical symmetry, the net charge must be uniform:

$\sigma_{tot}4\pi R^2 = Q_{tot} = 4\pi\epsilon_0 R V_0$ (since $V_0 = Q_{tot}/4\pi\epsilon_0 R$) $\Rightarrow \sigma_{tot} = \epsilon_0 V_0/R$. Thus

$$\sigma_f = \begin{cases} \epsilon_0 V_0/R & \text{on northern hemisphere (Note } \sigma_b = 0\text{, so } \sigma_b + \sigma_f = \sigma_{tot}) \\ (\epsilon_0 V_0/R)(1+\chi_e) & \text{on southern hemisphere (Note } \sigma_b = -\epsilon_0\chi_e V_0/R\text{, so } \sigma_b + \sigma_f = \sigma_{tot}) \end{cases}$$

(b) By construction, $\sigma_{tot} = \sigma_b + \sigma_f = \epsilon_0 V_0/R$ is uniform. So the potential of a uniformly charged sphere is $V(r) = \frac{Q_{tot}}{4\pi\epsilon_0 r} = \frac{\sigma_{tot}4\pi R^2}{4\pi\epsilon_0 r} = \frac{\epsilon V_0}{R}\frac{R^2}{\epsilon_0 r} = V_0\frac{R}{r}$ ✓

P 4.36

(c) Since everything is consistent, and the B.C.'s ($V = V_0$ at $r = R$, $V \to 0$ as $r \to \infty$) are met, the uniqueness thm guarantees that the soln we constructed is the unique soln of the problem

(d) For fig. (b), everything goes through as before. But for fig (a), since $\underline{P}$ is not perpendicular to $\hat{\underline{n}}$ on the flat surface here, there will be bound charge on it, and so spoiling the symmetry