Name solutions (27 present)

Physics 322 - Electrodynamics II **Final exam** **2.30 – 4.25 pm, 7 June 2006**
Spring 2006 Instructor: David Cobden

Do not turn this page until the buzzer goes at 2.30. You have up to 1h55m: give your exam to the TA before 4.25.

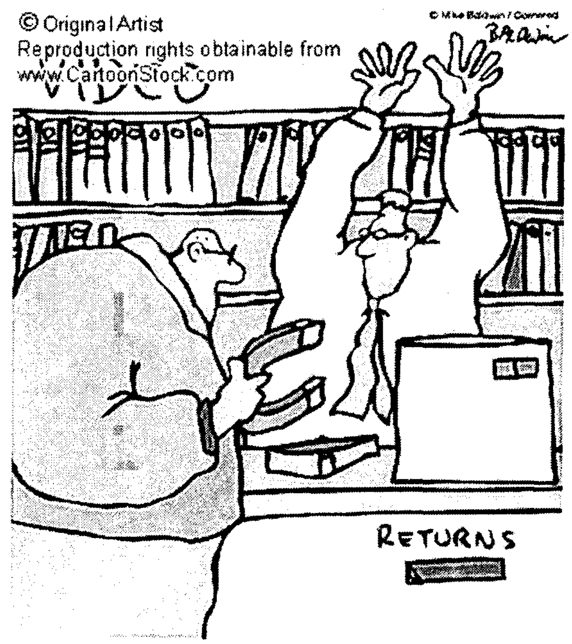
This exam contains 200 points. Be sure to attempt all the questions.

Please write your name on every page and your SID on the first page.

Write all your working on these question sheets. Use this front page for extra working. It is important to show your calculation or derivation. Some of the marks are given for showing clear, accurate and appropriate working and reasoning.

Watch the blackboard for corrections or clarifications during the exam.

This is a closed book exam. No books, notes or calculators are allowed.



Name solutions

1. [12] State Maxwell's equations with nonzero charge and current density.

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

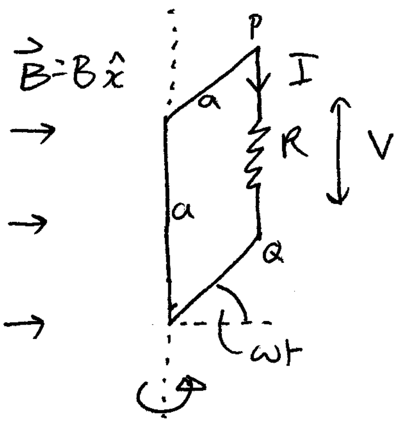
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t}$$

2. [8] State Faraday's law of induction and show how it leads to one of Maxwell's equations.

$$\mathcal{E} = -\frac{d\Phi}{dt} \quad \therefore \oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{S} = \int -\frac{\partial \vec{B}}{\partial t}\cdot d\vec{S}$$

$$\oint \vec{E}\cdot d\vec{l} \overset{\text{stokes}}{=} \int \vec{\nabla}\times\vec{E}\cdot d\vec{S} \quad \therefore \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

3. [15] A square wire loop of side *a* containing a resistor *R* rotates at angular frequency *ω* about one of its sides, which is the fixed rotation axis. A constant uniform magnetic field *B* is applied perpendicular to the axis. Find the power dissipated (Joule heating) in the resistor as a function of time *t*.



$$P = \frac{V^2}{R} \qquad V = \mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\left(Ba^2\sin\omega t\right)$$

$$= -Ba^2\omega\cos\omega t$$

$$\therefore P = \frac{B^2a^4\omega^2\cos^2\omega t}{R}$$

4. [15] By considering the torque, show that the answer to the previous question is equal to the power required to keep the loop rotating.

Only force on side PQ gives torque about axis:

$$\vec{N} = BIa\,\hat{y} \times a\cos\omega t\,\hat{x}$$

$$= -BIa^2\cos\omega t\,\hat{z} \qquad \left(\text{or } \vec{N} = \vec{m}\times\vec{B} = B.Ia^2.\cos\omega t.\hat{z}\right)$$

$$P = N\omega = B.\frac{V}{R}.a^2\cos\omega t.\omega$$

$$= B.\frac{Ba^2\omega\cos\omega t}{R}.a^2\omega\cos\omega t$$

$$= \frac{B^2a^4\omega^2\cos^2\omega t}{R}$$

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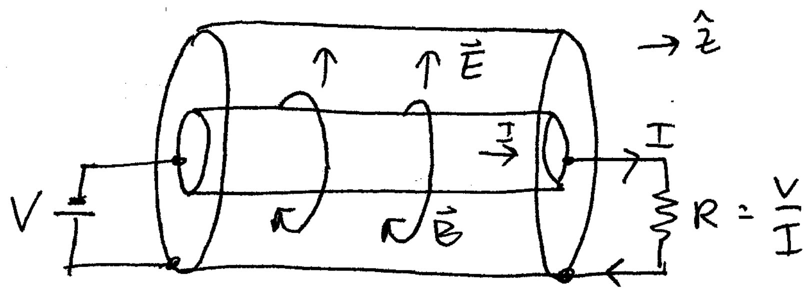
5. [5] Give an expression for the electromagnetic power flowing through a surface in terms of the **E** and **B** fields.

$$P = \int_S \frac{1}{\mu_0} \vec{E} \times \vec{B} \cdot d\vec{S}$$

6. [25] A coaxial cable has inner conductor radius a and outer conductor radius b. A battery of voltage V_0 is connected at one end between the inner and outer conductors, and a resistor R is connected at the other. Find the electric and magnetic fields within the cable (neglecting the resistance of the cable.)



$$\left.\begin{array}{l}\vec{E} = E(r)\hat{r} \\ \vec{B} = B(r)\hat{\phi}\end{array}\right\} a<r<b$$

zero otherwise

Ampere : $2\pi r B(r) = \mu_0 I \rightarrow B(r) = \frac{\mu_0 I}{2\pi r} = \frac{\mu_0 V}{2\pi r R}$

Gauss $\rightarrow E(r) = \frac{A}{r}$ ← const

To find A : $\int_b^a -E(r)\,dr = V = -A\int_b^a \frac{dr}{r} = A\ln\frac{b}{a} \quad \therefore A = \frac{V}{\ln b/a}$

$$\therefore E(r) = \frac{V}{r\ln b/a}$$

7. [15] Show that the electromagnetic power flowing along the cable is equal to the Joule heating in the resistor.

$$P = \int_a^b \int_0^{2\pi} \frac{1}{\mu_0} \vec{E}\times\vec{B} \cdot r\,d\phi\,dr\,\hat{z} = \frac{1}{\mu_0}\int_a^b E(r)B(r)\,2\pi r\,dr$$

$$= \frac{2\pi}{\mu_0}\int \frac{V}{r\ln b/a} \cdot \frac{\mu_0 I}{2\pi r} \cdot r\,dr = \frac{2\pi}{\mu_0} \cdot \frac{V}{\ln b/a} \cdot \frac{\mu_0 I}{2\pi} \cdot \int_a^b \frac{dr}{r}$$

$$= VI = \frac{V^2}{R}$$

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8. [15] For a monochromatic plane wave (MPW) with wavevector **k** and frequency ω, use Maxwell's equations to show that **E**, **B** and **k** are mutually perpendicular.

MPW: $\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot r - \omega t)}$ $\vec{B} = \vec{B}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

$\vec{\nabla} \to i\vec{k}$ $\therefore$ $\vec{\nabla}\cdot\vec{B} = 0 \longrightarrow i\vec{k}\cdot\vec{B} = 0 \Rightarrow \vec{k} \perp r \vec{B}$

$\vec{\nabla}\cdot\vec{E} = 0 \longrightarrow i\vec{k}\cdot\vec{E} = 0 \Rightarrow \vec{k} \perp r \vec{E}$

in vac

$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \to i\vec{k}\times\vec{E} = i\omega\vec{B} \to \vec{B} = \frac{\vec{k}\times\vec{E}}{\omega}$

$\Rightarrow \vec{B} \perp r \vec{E}$

9. [5] The electric field of a particular MPW is given by $\mathbf{E} = \mathrm{Re}\left[E_0(\hat{\mathbf{x}} + i\hat{\mathbf{y}})e^{i(kz-\omega t)}\right]$. What kind of polarization does it have?

circular (left handed, E_y is $\frac{\pi}{2}$ ahead of E_x)

10. [10] Find the magnetic field for this MPW.

$$\vec{B} = \mathrm{Re}\left[\frac{\vec{k}\times\vec{E}_0}{\omega}(\hat{x} + i\hat{y})e^{i(kz-\omega t)}\right]$$

$$= \mathrm{Re}\left[\frac{E_0}{c}(\hat{y} - i\hat{x})e^{i kz - \omega t}\right]$$

11. [10] Find the time-averaged Poynting vector, and hence the pressure exerted by the wave when normally incident on a fully absorbing surface.

$$\langle\vec{S}\rangle = \left\langle\frac{1}{\mu_0}\vec{E}\times\vec{B}\right\rangle = \frac{1}{\mu_0}\hat{k}\left[\langle E_x B_y\rangle + \langle E_y B_x\rangle\right]$$

same (for each linearly polarized component)

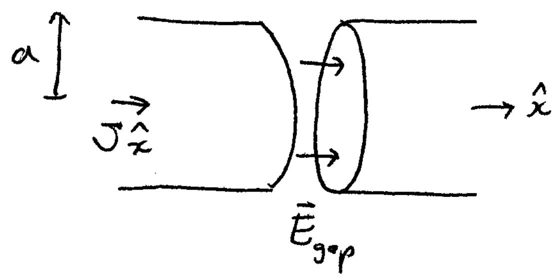
$$= \frac{2}{\mu_0}\hat{k}\left\langle E_0\cos(kz-\omega t)\,\frac{E_0}{c}\cos(kz-\omega t)\right\rangle$$

$$= \frac{2E_0^2}{\mu_0 c}\hat{k}\langle\cos^2\rangle = \frac{E_0^2}{\mu_0 c}\hat{k} = c\varepsilon_0 E_0^2\hat{k}$$

$$\text{Pressure} = \frac{\langle|\vec{S}|\rangle}{c} = \varepsilon_0 E_0^2$$

12. [25] A cylindrical wire of radius a carries an oscillating current $I = I_0 \cos \omega t$. On close inspection it is found to be broken by a cylindrical gap of length $b << a$. Find the displacement current density inside the gap (neglecting edge effects) and show that the total displacement current crossing the gap is equal to I.

Approximate // plate capacitor: $\vec{E}_{gap} = \frac{\sigma \hat{x}}{\varepsilon_0}$ $\quad \sigma = \int J\,dt = \int \frac{I}{\pi a^2}\,dt$



$$\vec{J}_d = \varepsilon_0 \frac{\partial \vec{E}_{gap}}{\partial t} = \frac{d\sigma}{dt}\hat{x} = J\hat{x} = \frac{I}{\pi a^2}\hat{x} = \frac{I_0 \cos \omega t}{\pi a^2}\hat{x}$$

$$\int_{\substack{\text{surface} \\ \text{thr' gap}}} \vec{J}_d \cdot d\vec{S} = \pi a^2 J_d = I$$

13. [10] Find the magnetic field inside the gap.

$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_d$ in gap $(\vec{J} = 0)$

Symmetry $\rightarrow$ $\vec{B} = B(r)\,\hat{\phi}$

$$\oint_{\substack{\text{coaxial} \\ \text{circle}}} \vec{B} \cdot d\vec{l} = \mu_0 \int_{\text{circle}} \vec{J}_d \cdot d\vec{S}$$

$\therefore 2\pi r B(r) = \mu_0 \pi r^2 J_d$ $\qquad \therefore B(r) = \frac{\pi r^2 \mu_0}{2\pi r} \cdot \frac{I}{\pi a^2} = \frac{\mu_0 r I}{2\pi a^2}$

$$= \frac{\mu_0 r I_0 \cos \omega t}{2\pi a^2}$$

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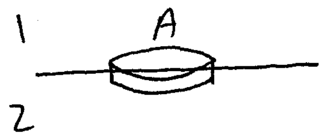
14. [10] Using one of Maxwell's equations, derive the boundary condition for $B_\perp$ at an interface between two materials, and say whether you think this boundary condition is ever violated.



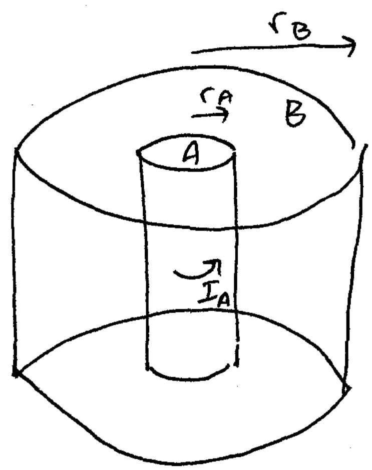
Consider pillbox:

$$\int \vec{B}.d\vec{S} = (B_{1\perp} - B_{2\perp})A$$

$$\text{Gauss} \downarrow = \int \vec{\nabla}.\vec{B}\, d^3r = 0 \qquad \therefore\ B_{1\perp} = B_{2\perp}$$

Only untrue if $\vec{\nabla}.\vec{B} \neq 0$, ie monopoles exist.

15. [20] Two long solenoids, A and B, are coaxial. They have the same length l and number of turns N but different radii, $r_A < r_B$. Find the mutual inductance between them.



$$M = \frac{\Phi_{BA}}{I_A}$$

where Φ_{BA} = flux linked by B when I_A flows in A

$$\text{then } \overset{\text{(field)}}{B_m} = 0 \quad r > r_A$$

$$= \frac{\mu_0 N I_A}{L} \quad r < r_A$$

$$\therefore\ \Phi_{BA} = N.\pi r_A^2.\frac{\mu_0 N I_A}{L} = \frac{\mu_0 \pi r_A^2 N^2 I_A}{L}$$

$$\therefore\ M = \frac{\mu_0 \pi r_A^2 N^2}{L}$$