322 MidTerm 2
Solutions

1. (a) In terms of *free* charge and current densities, $\rho_f, \mathbf{J}_f$ state the partial differential equation version of Maxwell's equations.

$$\vec{\nabla} \cdot \vec{D} = \rho_f, \quad \vec{\nabla} \cdot \vec{B} = 0$$

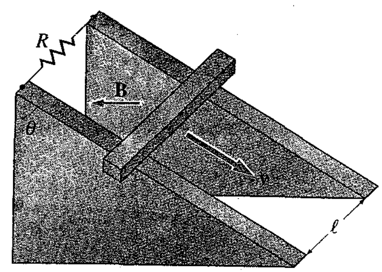
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

(b) Use Maxwell's equations to show that $\nabla \cdot \mathbf{J}_f + \frac{\partial \rho_f}{\partial t} = 0$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = 0 = \vec{\nabla} \cdot \vec{J}_f + \vec{\nabla} \cdot \frac{\partial \vec{D}}{\partial t} = \vec{\nabla} \cdot \vec{J}_f + \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{D}$$

$$= \vec{\nabla} \cdot \vec{J}_f + \frac{\partial \rho}{\partial t}$$

(c) A conducting bar of mass m, slides down the conducting wedges shown in the figure. The wedges are separated by a distance l, connected at the top by a resistance R and make an angle θ with the vertical. A uniform magnetic field, $\mathbf{B}$ points horizontally as shown. The bar is released from rest and slides down the rails, moving with speed v such that $|\frac{\partial \Phi}{\partial t}| = BLv\cos\theta$. Determine the terminal speed (final constant speed) of the bar.

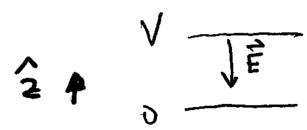


downward force of gravity along wedge $= mg\cos\theta$ is balanced by magnetic force $I\ell B\cos\theta$

$$IR = \left|\frac{\partial \Phi}{\partial t}\right| = B\ell v\cos\theta$$

so $$mg\cos\theta = \frac{(B\ell)^2 v\cos\theta\cos\theta}{R}$$

$$v = \frac{mgR}{(B\ell)^2\cos\theta}$$



2. A capacitor with parallel plates, with radius a and separation $d \ll a$ has a potential difference $V(t) = V_0 \frac{t}{\tau_0}$, which is the potential at the top ($z = d$) minus that of the bottom ($z = 0$), and τ_0 is a constant with dimensions of time.

(a) Show that the magnetic field $\mathbf{B}$ obeys the equation: $\nabla \times \mathbf{B}(s, z, t) = -\frac{\mu_0 \epsilon_0}{\tau_0 \, d} V_0 \widehat{\mathbf{k}}$, where s is the distance from the axis of symmetry.

$\vec{E} = -\frac{V(t)}{d} \hat{k}$

$$\vec{\nabla} \times \vec{B} = + \frac{\partial \vec{E}}{\partial t} \mu_0 \varepsilon_0$$
$$= \frac{\mu_0 \varepsilon_0}{d} \left(+ \frac{V_0}{\tau} \right) \hat{k}$$

(b) Determine $\mathbf{B}$ for $0 < z < d$ and $s < a$. $\vec{B}$ is along $\hat{\phi}$ direction

$\vec{B} = B \hat{\phi}$

$$\oint_C \vec{B} \cdot d\vec{l} = \varepsilon_0 \frac{\mu_0 V_0}{d \tau_0} \text{ area}$$

$$2\pi B s = \varepsilon_0 \frac{\mu_0 V_0}{d \tau_0} \pi s^2 \Rightarrow \boxed{\vec{B} = \varepsilon_0 \frac{\mu_0 V}{d \tau_0} \frac{s}{2} \hat{\phi}}$$

(c) Determine $\mathbf{B}$ for $0 < z < d$ and $s > a$.

$\vec{B}$ is along $\hat{\phi}$ direction

$\vec{B} = B \hat{\phi}$

$$\oint \vec{B} \cdot d\vec{l} = \varepsilon_0 \frac{\mu_0 V_0}{d \tau_0} \text{ area}$$

$$B = \frac{1}{2\pi s} \varepsilon_0 \frac{\mu_0 V_0}{d \tau_0} \pi a^2$$

$$\vec{B} = \frac{\mu_0 \varepsilon_0 V_0}{\tau_0 d} \frac{a^2}{2s} \hat{\phi}$$

3. An infinitely long solenoid of radius R, n turns per unit length, and current I_0 lies along the z axis, which also is the axis of the solenoid. $\mathbf{B} = \mu_0\, nI\, \hat{\mathbf{k}}$. Let s be the distance away from the z axis.

(a) Determine the magnetic energy density for $s < R$. .

$$u = \frac{1}{2\mu_0} B^2 = \frac{\mu_0}{2} n^2 I^2$$

(b) Now suppose that the current is time-dependent, so that, if $t > 0$, it decays slowly with time: $I(t) = I_0 e^{-t/\tau_0}$. Determine $\mathbf{E}(s,t)$ for positions inside the solenoid, where s is the distance away from the center. $\vec{E} = E\,\hat{\phi}$ $\quad \oint_C \vec{E}\cdot d\vec{l} = -\frac{\partial \Phi}{\partial t}$ $\quad$ C is circle of radius s

$$2\pi s E = -\pi s^2 \mu_0 n \frac{\partial I}{\partial t} = \pi s^2 \frac{\mu_0 n I_0}{\tau_0} e^{-t/\tau_0}$$

$$\vec{E} = \frac{s}{2} \frac{\mu_0 n I_0}{\tau_0} e^{-t/\tau_0}\, \hat{\phi}$$

(c) Determine the density of electromagnetic momentum and the vector $\nabla\cdot \overleftrightarrow{\mathbf{T}}$.

$$\vec{\wp}_{em} = \epsilon_0 (\vec{E} \times \vec{B}) = \epsilon_0 \frac{(\mu_0 n)^2 I^2(t)}{\tau_0} \frac{s}{2} (\hat{\phi} \times \hat{k})$$

$$= \epsilon_0 \frac{(\mu_0 n)^2 I^2(t)}{\tau_0} \frac{\vec{s}}{2}$$

$$\vec{\nabla} \cdot \overleftrightarrow{T} = \frac{\partial \vec{\wp}_{em}}{\partial t} = -\frac{\epsilon_0 (\mu_0 n)^2}{\tau_0^2} \vec{s}\, I_0^2 e^{-2t/\tau_0}$$