 Name solutions

Electromagnetism, Physics 322 **Winter 2004**

First midterm Instructor: David Cobden

8.20 am, January 30, 2004

You have 60 minutes. End on the buzzer. Answer all 13 questions.

Write your name on every page and your ID on the first page.

Write all your working on these question sheets. Use this cover page for extra working (you might get credit for it.)

It is important to show your calculation or derivation. You won't get full marks just for stating the correct answer if you don't show how you get it.

Watch the blackboard for corrections or clarifications during the exam.

This is a **closed book** exam. **No notes allowed. No calculators.**

Standard notation for spherical coordinates is used throughout. Thus, for example, r is always the distance from the origin. We also use r for the distance from the exis in cylindrical coordinates.

Do not turn this page until I say 'go'!

We write Cartesian unit vectors according to the form $\mathbf{x} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.

In cylindrical coordinates, $\mathbf{x} = (r, \phi, z)$, $\hat{\mathbf{r}} \times \hat{\boldsymbol{\varphi}} = \hat{\mathbf{k}}$,

and
$$\nabla \times \mathbf{A} = \left[\frac{1}{r}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right]\hat{\mathbf{r}} + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]\hat{\boldsymbol{\varphi}} + \frac{1}{r}\left[\frac{\partial (r A_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right]\hat{\mathbf{z}} .$$

The field of a dipole $\mathbf{m}$ is given by $\mathbf{B} = \frac{\mu_0}{4\pi r^3}[3(\mathbf{m}.\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}]$.

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Part I. A solid, nonmagnetic, cylindrical wire of radius R centered on the z-axis carries a current density $\mathbf{J}(\mathbf{x}) = \alpha r^2 \hat{\mathbf{k}}$, where r is distance from the axis.

1. [6] State Ampere's law in differential form and derive from it the integral form relating **B** and $I_{enc.}$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\therefore \int \vec{\nabla} \times \vec{B} \cdot d\vec{S} = \int \mu_0 \vec{J} \cdot d\vec{S} \quad \text{over any surface } S$$

$$\therefore \oint \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{S} \quad \text{by Stokes, where } C \text{ surrounds } S$$

$$= \mu_0 I \quad I = \text{total current through } S$$

2. [8] Find **B** outside the wire.

Cylindrical symmetry $\rightarrow \vec{B} = \vec{B}(r)$. Current reversal symmetry $\rightarrow = B_\phi(r)\hat{\phi}$ everywhere.

Amperian circular loop enclosing wire:

$$\int \vec{B} \cdot d\vec{l} = 2\pi r B_\phi = \mu_0 \int_0^R J . 2\pi r dr = 2\pi \mu_0 \alpha \int_0^R r^3 dr = 2\pi . \frac{\mu_0 \alpha R^4}{4}$$

$$\therefore B_\phi = \frac{\mu_0 \alpha R^4}{4r} \quad \text{for } r > R$$

3. [8] Find **B** inside the wire.

Amperian loop within wire:

$$2\pi r B_\phi = \mu_0 \int_0^r \alpha r'^2 \, 2\pi r' dr' = 2\pi . \frac{\mu_0 \alpha r^4}{4}$$

$$\therefore B_\phi = \frac{\mu_0 \alpha r^3}{4} \quad \text{for } r < R$$

4. [4] Find **H** inside the wire.

$$\vec{H} = \frac{1}{\mu_0}\vec{B} + \vec{M} \qquad \vec{M} = 0 \text{ (nonmagnetic) everywhere}$$

$$\therefore \vec{H} = \frac{1}{\mu_0}\vec{B} \text{ everywhere.}$$

$$\therefore \vec{H} = \frac{\alpha r^3}{4}\hat{\phi} \quad \text{for } r < R$$

5. [10] Find **A** outside the wire, in the Coulomb gauge. Take **A** to vanish at the surface of the wire.

$$\vec{\nabla} \times \vec{A} = \vec{B} = \frac{\mu_0 \alpha R^4}{4r}\hat{\phi} \quad \text{for } r > R$$

Cylindrical symmetry $\rightarrow$ $\vec{A} = \vec{A}(r)$

in Coulomb gauge $= A_z(r)\hat{k}$ because $\vec{A} \sim \int \frac{\vec{J}(\vec{x}')d^3x'}{|\vec{x}-\vec{x}'|} \propto \hat{k}$

$$\vec{\nabla} \times \vec{A} = -\frac{\partial A_z}{\partial r}\hat{\phi} \quad \text{(from cover page)}$$

$$\therefore -\frac{\partial A_z}{\partial r} = \frac{\mu_0 \alpha R^4}{4r} \qquad \therefore A_z = -\frac{\mu_0 \alpha R^4}{4}\ln r + \text{const}$$

If $A_z = 0$ at $r = R$ then $\vec{A} = -\frac{\mu_0 \alpha R^4}{4}\ln \frac{r}{R}\,\hat{k}$

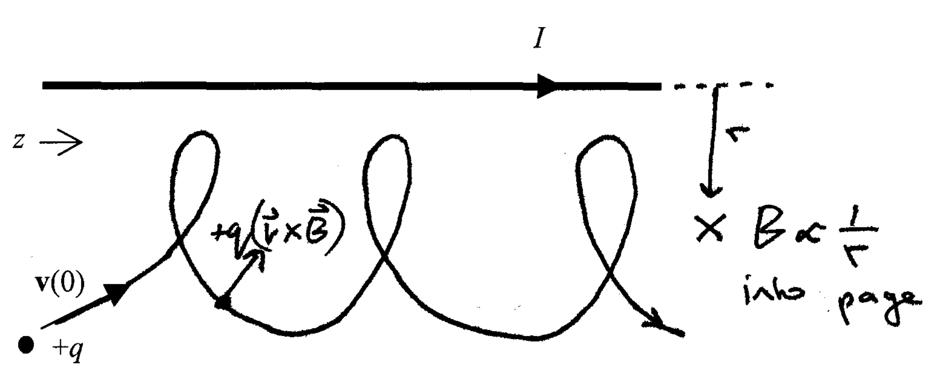
6. [8] At time $t = 0$ a particle with charge $+q$ has a velocity coplanar with the wire, $\mathbf{v}(0) = v_r(0)\,\hat{\mathbf{r}} + v_z(0)\,\hat{\mathbf{k}}$. Subsequently the particle *comes close to the wire but never collides with it.* What can you say about its velocity $\mathbf{v}(t)$ at all later times?

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) = q\vec{v} \times \vec{B} \quad \text{because } \vec{E} = 0.$$

(a) $\vec{F}$ is always perpendicular to $\vec{v}$, so $|\vec{v}|$ is conserved.

(b) $\vec{F}$ is in the plane of $\vec{v}$ and the wire, so $\vec{v}$ remains coplanar with the wire.

7. [6] Sketch below the motion of the particle if $v_z(0) > 0$:



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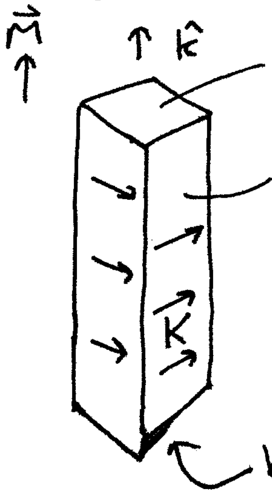
Part II. A rectangular bar magnet has a square cross-section of side W in the x-y plane, a length $L >> W$ in the z-direction, and uniform magnetization $\mathbf{M} = M\hat{\mathbf{k}}$ within it.

8. [10] Show that the magnetic field $\mathbf{B}(\mathbf{x})$ produced by the magnet is identical to that produced by a rectangular solenoid.



ends : $\vec{K}_b = \vec{M} \times \hat{n} = \vec{M} \times \hat{k} = 0$

side : $\vec{K}_b = M\hat{k} \times \hat{n}$
$= M$ in direction across face perpendicular to axis.

$\therefore$ Looks like a square cross-section solenoid with surface current density $K = K_b = M$.

bulk: $\vec{J}_b = \vec{\nabla} \times \vec{M} = 0$

9. [6] Hence find $\mathbf{B}(\mathbf{x})$ inside the magnet far from the ends.

For a solenoid, $\vec{B} = \mu_0 n I \hat{k}$ $\quad$ n = turns/unit length

here, $nI = K = K_b = M$ $\quad$ $\therefore \vec{B} = \mu_0 M \hat{k} = \mu_0 \vec{M}$.

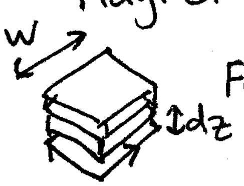
10. [6] Give an expression, involving a surface integral, for the magnetic dipole moment of a current loop which carries current I along a closed curve C in three-dimensional space.

$\vec{m} = I \int_S d\vec{S}$ where S = any surface bounded by C.

$= I\vec{S}$ ← vector area of loop

11. [10] By using this expression applied the bound current density, find the total magnetic dipole moment $\mathbf{m}$ of the bar magnet. Show also that the same value of $\mathbf{m}$ can be obtained in a more direct way.

Magnet looks like a stack of square current loops.



For loop of height dz, $I = K dz$

$d\vec{m} = I\vec{S} = IW^2\hat{k} = K dz W^2 \hat{k}$

$\therefore \vec{m} = \int d\vec{m} = \int_0^L K W^2 dz \hat{k} = W^2 L K \hat{k} = W^2 L \vec{M}$

Naturally this equals $\int_{\text{magnet volume}} \vec{M} d^3x$

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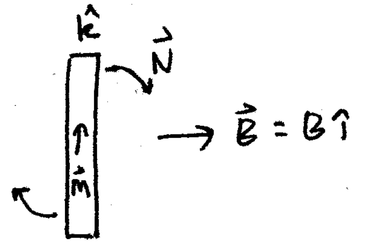
12. [8] The bar magnet is free to move, but is initially stationary. A uniform magnetic field perpendiculat to its axis, $\mathbf{B} = B\hat{\mathbf{i}}$ is then switched on. What happens?

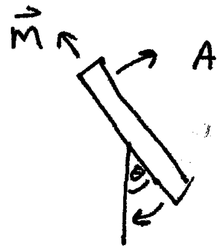


Net force $\vec{F} = -\vec{\nabla}(-\vec{m}\cdot\vec{B}) = 0$ (constant)

$\therefore$ no translational motion.

Net torque $\vec{N} = \vec{m} \times \vec{B} = mB\hat{\jmath}$

$\therefore$ magnet rotates as shown.



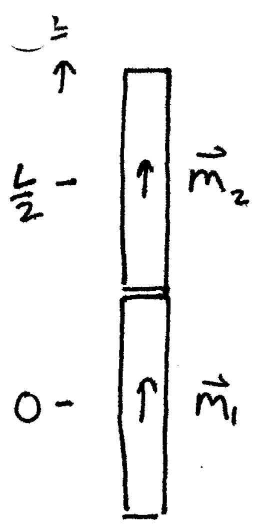
At angle θ,

$\vec{N} = mB\sin\theta\,\hat{\jmath}$

Equation of motion: $I\dfrac{\partial^2\theta}{\partial t^2} = mB\sin\theta$ (moment of inertia)

θ oscillates between 0 and π (not harmonic!)

13. [10] The bar magnet is sawn into two equal parts each of length $L/2$, which stick end-to-end. *Estimate roughly* the force of attraction between the two parts. (Assume no externally applied field.)



Approximate as force between two dipoles

$\vec{m}_2 = m\hat{k}$ at $z = \frac{L}{2}$

and $\vec{m}_1 = m\hat{k}$ at $z = 0$.

$\vec{F} = -\vec{\nabla}U$

$$U = -\vec{m}_2\cdot\vec{B}_1 = -m\hat{k}\cdot\left\{\frac{\mu_0}{4\pi z^3}\left[3(\vec{m}_1\cdot\hat{r})\hat{r} - \vec{m}_1\right]\right\}$$

$$= -\frac{m\mu_0}{4\pi z^3}\cdot 2m = -\frac{\mu_0 m^2}{2\pi z^3}$$

$$\vec{F} = -\frac{\partial U}{\partial z}\bigg|_{z=L}\hat{k} = \frac{\partial}{\partial z}\left(\frac{\mu_0 m^2}{2\pi z^3}\right)\bigg|_{z=L}\hat{k} = -\frac{3\mu_0 m^2}{2\pi z^4}\bigg|_{z=L}\hat{k}$$

$$= -\frac{3\mu_0\left(W^2\frac{L}{2}M\right)^2}{2\pi\left(\frac{L}{2}\right)^4}\hat{k} = \frac{-3\cdot 2^4}{2\cdot 2^2}\cdot\frac{\mu_0 W^4 L^2 M^2}{L^4}\hat{k}$$

$\therefore$ attractive force $= 6\mu_0\dfrac{W^4}{L^2}M^2$