Name solutions

Electromagnetism, Physics 322
Winter 2004

Final exam
Instructor: David Cobden

8.20 am, March 16, 2004

Start on the buzzer at 8.20. You have 120 minutes. End on the buzzer at 10.20. Answer all questions.

Write your name on each page and your student ID on the first page.

Write all your working on these question sheets. Use this cover page for extra working (you might get credit for it.)

It is important to show your calculation or derivation. You won't get full marks just for stating the correct answer if you don't show how you get it.

Watch the blackboard for corrections and clarifications during the exam.

This is a **closed book** exam. **No notes. No calculators.**

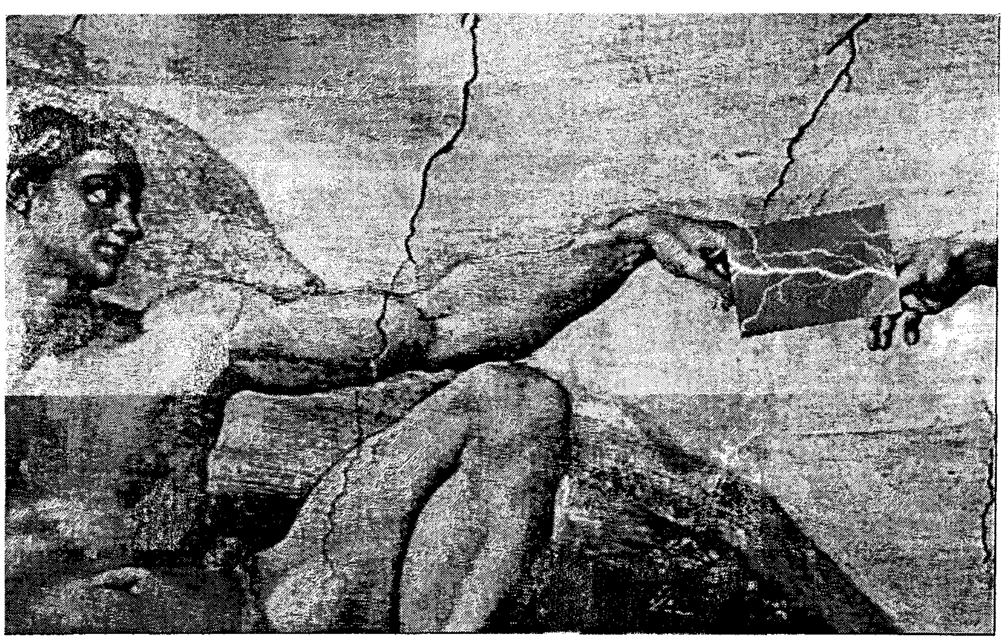
Do not turn this page until I say 'go'.

We write Cartesian unit vectors according to the form $\mathbf{x} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.

In cylindrical coordinates, $\mathbf{x} = (r, \phi, z)$, $\hat{\mathbf{r}} \times \hat{\boldsymbol{\varphi}} = \hat{\mathbf{k}}$,

and $$\mathbf{B} = \nabla \times \mathbf{A} = \left[\frac{1}{r}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right]\hat{\mathbf{r}} + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]\hat{\boldsymbol{\varphi}} + \frac{1}{r}\left[\frac{\partial (rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right]\hat{\mathbf{k}}.$$

Poynting vector: $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$



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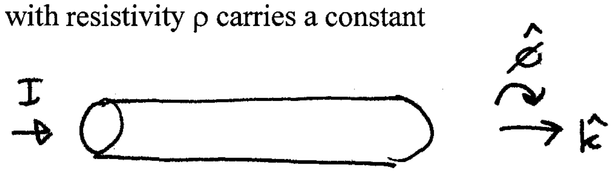
I. A long, solid, cylindrical wire of radius *a* made of a material with resistivity ρ carries a constant current *I.*



1. [5] What is the electric field within the wire?

$$\vec{E} = \rho J \hat{k} = \frac{\rho I}{\pi a^2}\hat{k} \text{ along the wire.}$$

2. [10] Find the magnetic field within the wire (ie, for $r < a$).

$$\vec{B} = B_\phi(r)\hat{\phi}$$

$$\oint_{\substack{\text{circle}\\\text{about}\\\text{axis}}} \vec{B}\cdot d\vec{l} = \mu_0 \pi r^2 J = \mu_0 I \frac{r^2}{a^2} \qquad \therefore\ 2\pi r B_\phi = \mu_0 I \frac{r^2}{a^2}$$

$$\vec{B} = \frac{\mu_0 I r}{2\pi a^2}\hat{\phi}$$

3. [20] Show that the flow of electromagnetic energy inwards through a cylinder of radius $r < a$ and length *l*, coaxial with and internal to the wire, is equal to the Joule heating within that cylinder.

Inside wire, $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}\left(\frac{\rho I}{\pi a^2}\hat{k}\right)\times\left(\frac{\mu_0 I r}{2\pi a^2}\hat{\phi}\right)$

$$= \frac{\rho I^2 r}{2\pi^2 a^4}\hat{r}$$

$\therefore$ Flow of energy in through cylinder is

$$\frac{dW}{dt} = -\int_{\text{cylinder}} \vec{S}\cdot d\vec{A} = \frac{\rho I^2 r}{2\pi^2 a^4}\cdot 2\pi r L = \frac{\rho I^2 r^2 L}{\pi a^4}$$

Joule heating is $\int_{\text{cylinder}} \vec{E}\cdot\vec{J}\, d^3x = \frac{\rho I}{\pi a^2}\cdot\frac{I}{\pi a^2}\cdot \pi r^2 L$

$$= \frac{\rho I^2}{\pi a^4} r^2 L = \frac{dW}{dt} \qquad \text{QED}$$

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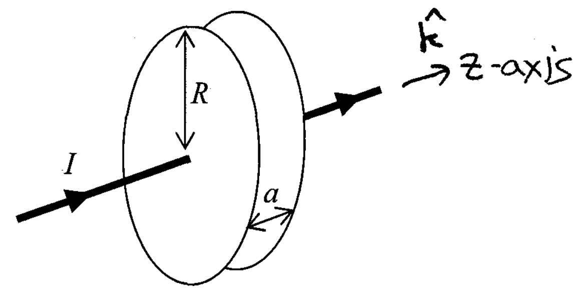
II. A circular capacitor is connected between two thin straight wires as shown. The radius of the plates is R, and their separation is $a << R$, so that fringing effects may be neglected. A current I is passed along the wires.



4. [10] State the Maxwell's equation which relates **H**, **D** and $\mathbf{J}_f$ (in differential form). Write the same equation also in terms of the displacement current density, $\mathbf{J}_D = \frac{\partial \mathbf{D}}{\partial t}$.

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}_f = \vec{J}_D + \vec{J}_f$$

5. [10] Show that the magnitude of the electric field between the plates is determined by $\frac{\partial E}{\partial t} = \frac{I}{\varepsilon_0 \pi R^2}$.

$E = \frac{\sigma}{\varepsilon_0}$ Conservation of charge $\rightarrow$ $I = \frac{d}{dt}\pi R^2 \sigma = \pi R^2 \frac{d\sigma}{dt}$

$= \pi R^2 \varepsilon_0 \frac{dE}{dt}$

$$\therefore \frac{\partial E}{\partial t} = \frac{I}{\pi R^2 \varepsilon_0}$$

6. [10] Hence show that $\mathbf{J}_D = \frac{I}{\pi R^2}\hat{k}$ between the plates.

$$\vec{J}_D = \frac{\partial \vec{D}}{\partial t} = \frac{\partial \vec{E}}{\partial t} = \frac{I}{\pi R^2}\hat{k}$$

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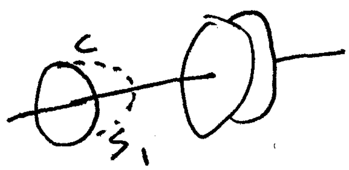
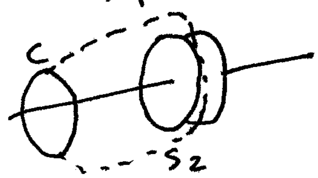
7. [10] Write down the integral version of the latter equation in Q.4, relating a line integral of **H** to a surface integral. Use the result of Q.6 to illustrate the validity of this equation by applying it to two different surfaces, each bounded by the same loop C encircling one of the wires, through which the total real current is not the same.

$$\oint_C \vec{H}\cdot d\vec{l} = \int_S \left(\vec{J}_D + \vec{J}_f\right)\cdot d\vec{A}$$ where $S =$ any surface bounded by C



surface S_1 intersects wire:

$$\text{RHS} = \int_{S_1} \left(0 + \vec{J}_f\right)\cdot d\vec{A} = I$$



surface S_2 passes between plates:

$$\text{RHS} = \int_{S_2} \left(\vec{J}_D + 0\right)\cdot d\vec{A} = \left(\frac{I}{\pi R^2}\right)\pi R^2 = I \text{ again.}$$

8. [10] Find $\mathbf{B} = \mu_0 \mathbf{H}$ between the plates (for $r < R$).

$$\vec{\nabla}\times\vec{H} = \frac{\partial \vec{D}}{\partial t} = \frac{I}{\pi R^2}\hat{k}$$ between plates, $\vec{H} = H(r)\hat{\phi}$

$$\therefore \oint \vec{H}\cdot d\vec{l} = \frac{I}{\pi R^2}\cdot \pi r^2 = \frac{I r^2}{R^2} \quad \therefore H(r) = \frac{I r}{2\pi R^2}$$

$$= 2\pi r H(r)$$

$$\therefore \vec{B} = \mu_0 \vec{H} = \frac{\mu_0 I r}{2\pi R^2}\hat{\phi}$$

9. [15] Find the vector potential $\mathbf{A}(r,t)$ between the plates, in the Coulomb gauge.

$$\vec{\nabla}\times\vec{A} = \vec{B}$$ Coulomb gauge, $\vec{A} \sim \int \frac{\vec{J}}{r} d^3x$

+ symmetry $\rightarrow \vec{A} = A_z(r)\hat{k}$

$$\therefore \vec{\nabla}\times\vec{A} = -\frac{\partial A_z}{\partial r}\hat{\phi} \qquad \therefore \frac{\partial A_z}{\partial r} = -\frac{\mu_0 I r}{2\pi R^2}$$

$$\therefore A_z = -\frac{\mu_0 I r^2}{4\pi R^2} + \text{const}_t$$

Take $\vec{A} = 0$ at $r = 0$ $\quad \therefore \vec{A} = -\frac{\mu_0 I r^2}{4\pi R^2}\hat{k}$

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III. A particular laser beam may be treated as a linearly polarized plane wave, whose electric field is given by $\mathbf{E} = E_0\hat{\mathbf{j}}\cos(\mathbf{k}.\mathbf{x} - \omega t)$, where $\mathbf{k} = k\hat{\mathbf{k}} = (0,0,k)$ in Cartesian coordinates.

10. [12] State the four Maxwell's equations (in partial differential form), relating **E** and **B** to the total charge density ρ and current density **J**.

$$\vec{\nabla}.\vec{E} = \rho/\varepsilon_0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}.\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t}$$

11. [10] Use them, together with the identity $\nabla\times(\nabla\times\mathbf{B}) = \nabla(\nabla.\mathbf{B}) - \nabla^2\mathbf{B}$, to show that in vacuum **B** obeys a three-dimensional wave equation with wave speed $c = \dfrac{1}{\sqrt{\varepsilon_0\mu_0}}$.

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\times\Big(\underbrace{\mu_0\vec{J}}_{0} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\Big)$$

$$\therefore \vec{\nabla}(\underbrace{\vec{\nabla}.\vec{B}}_{0}) - \nabla^2\vec{B} = \mu_0\varepsilon_0\,\vec{\nabla}\times\frac{\partial\vec{E}}{\partial t}$$

$$= \mu_0\varepsilon_0\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

$$\therefore -\nabla^2\vec{B} = \mu_0\varepsilon_0\frac{\partial}{\partial t}\left(-\frac{\partial\vec{B}}{\partial t}\right) = -\mu_0\varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2}$$

$$\therefore \nabla^2\vec{B} = \mu_0\varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} \qquad c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$$

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12. [8] What is **B** in the laser beam?

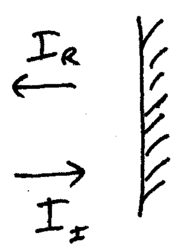
$$\vec{B} = \frac{\vec{k}\times\vec{E}}{\omega} = \frac{E_0}{c}\,\hat{k}\times\hat{j}\cos(\vec{k}\cdot\vec{x}-\omega t)$$
$$= -\frac{E_0}{c}\,\hat{i}\cos(\vec{k}\cdot\vec{x}-\omega t)$$

13. [10] Show, using the Poynting vector, that the *time-averaged* power flowing per unit cross-sectional area of the beam is $\frac{\varepsilon_0 E_0^{\,2} c}{2}$.

This is the intensity, $I = \langle S \rangle = \langle \varepsilon_0 E^2 c \rangle$

$$= \langle \varepsilon_0 E_0^2 \cos^2(\vec{k}\cdot\vec{x}-\omega t)\, c \rangle$$
$$= \varepsilon_0 E_0^2 c \,\langle \cos^2(\vec{k}\cdot\vec{x}-\omega t) \rangle$$
$$= \frac{1}{2}\varepsilon_0 E_0^2 c$$

14. [10] The laser beam is incident normally on a flat surface, where it is partially absorbed and partially reflected. The reflected beam has electric field amplitude αE_0 (where $\alpha < 1$). What is the average pressure exerted on the surface?



pressure $p = \frac{I_I + I_R}{c} = \frac{(1+\alpha^2)\frac{1}{2}\varepsilon_0 E_0^2 c}{c}$

$$= \frac{(1+\alpha^2)}{2}\varepsilon_0 E_0^2$$

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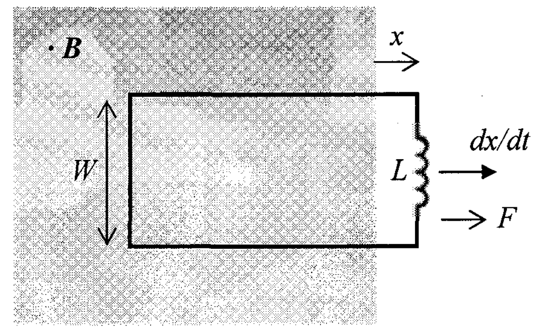
IV. Last question … A rectangular loop of wire lies initially in a region (shaded) where the magnetic field perpendicular to its plane is B. Outside the shaded region the magnetic field is zero. The loop has zero resistance but includes an inductor of self-inductance L, as shown. The loop is pulled to the right, such that after a time t the right hand side protrudes from the edge of the field region by distance x, as indicated. Take the mass of the loop to be negligible, and assume that there is no mutual inductance between the inductor and the loop.



15. [20] Show that the force required to pull the loop is given by $F = \dfrac{W^2B^2x}{L}$

$$F = BIW$$

Emf $\varepsilon = -\dfrac{d\Phi}{dt} = +BW\dfrac{dx}{dt}$

must be balanced by voltage across inductor

so $+BW\dfrac{dx}{dt} = L\dfrac{dI}{dt}$ $\quad\therefore BWx = LI \quad (I=0 \text{ at } x=0)$

$$\therefore F = BW.\frac{BWx}{L} = \frac{B^2W^2x}{L}$$