 Name __________solutions_______________

Electrodynamics, Physics 322 **Final exam** **8.20 am, March 15, 2005**
Winter 2005 Instructor: David Cobden

You have 120 minutes. End on the buzzer at 10.20. Answer all questions.

Write your name on every page and your ID on the first page.

Write all your working on these question sheets. Use this cover page for extra working (you might get credit for it.)

It is important to show your calculation or derivation. You won't get full marks just for stating the correct answer if you don't show how you get it.

Watch the blackboard for corrections or clarifications during the exam.

This is a **closed book** exam. **No notes allowed. No calculators!**

Do not turn this page until the buzzer goes!

energy density in monochromatic plane wave: $\langle u_{em} \rangle = \dfrac{e_0 E_0^{\;2}}{2}$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla.\mathbf{A}) - \nabla^2 \mathbf{A} \qquad \nabla.\mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial}{\partial f}A_f + \frac{\partial}{\partial z}A_z$$

from Nearingzero.com

I. Maxwells and monopoles

1. [15] State Maxwell's equations.

(i) $\vec{\nabla}.\vec{E} = \rho/\varepsilon_0$ (iii) $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

(ii) $\vec{\nabla}.\vec{B} = 0$ (iv) $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$

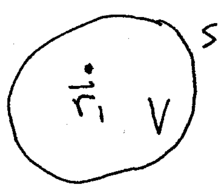
2. [10] Show that Maxwell's equations guarantee conservation of electric charge.

$\vec{\nabla}.(iv) \rightarrow \vec{\nabla}.(\vec{\nabla}\times\vec{B}) = \mu_0\vec{\nabla}.\vec{J} + \mu_0\varepsilon_0\vec{\nabla}.\frac{\partial\vec{E}}{\partial t}$

$= 0 \quad = \mu_0\vec{\nabla}.\vec{J} + \mu_0\varepsilon_0\frac{\partial}{\partial t}(\vec{\nabla}.\vec{E}) = \mu_0\vec{\nabla}.\vec{J} + \mu_0\varepsilon_0\frac{\partial}{\partial t}\frac{\rho}{\varepsilon_0}$

$\therefore \vec{\nabla}.\vec{J} = -\frac{\partial\rho}{\partial t}$ continuity eqn for charge → cons. of charge

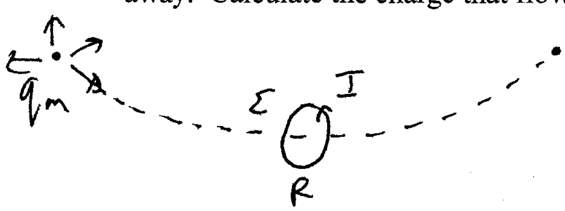
3. [10] A magnetic monopole at point $\mathbf{r}_1$ has charge q_m, such that $\nabla.\mathbf{B} = \mu_0 q_m \delta(\mathbf{r}-\mathbf{r}_1)$. Find the total magnetic flux through a surface surrounding the monopole ("zero because they don't exist" is not a valid answer.)



$\Phi_m = \oint \vec{B}.d\vec{S} = \int_V \vec{\nabla}.\vec{B}\, d^3r = \mu_0 q_m \int_V \delta(\vec{r}-\vec{r}_1)\, d^3r$

$= \mu_0 q_m$ if $\vec{r}_1$ is in V

4. [10] A ring of copper wire of resistance R is located near the origin. A magnetic monopole comes from somewhere far away, passes through the ring, and continues on its journey to somewhere else far away. Calculate the charge that flows around the ring during the transit of the monopole.



$I = \frac{\varepsilon}{R} = \frac{1}{R}\left(-\frac{d\Phi}{dt}\right)$

$Q = \int I\, dt = \frac{1}{R}\int -\frac{d\Phi}{dt}\, dt = \frac{-1}{R}\int d\Phi$

$= (-)\frac{1}{R}\Delta\Phi$ ← change in flux thr' ring

$= (-)\frac{1}{R} \times$ (number of B lines that have cut ring)

$= (-)\frac{1}{R}\Phi_m \quad \therefore |Q| = \frac{\mu_0 q_m}{R}$

5. [5] What happens if instead the ring is superconducting, so that $R = 0$?

No flux lines can "cut" the ring or the current would become infinite

$\therefore$ flux Φ_m is trapped through the ring

→ current I in ring changes (by $\Delta I = \frac{\Phi_m}{L}$ ← self inductance of ring)

II. Plane waves.

6. [10] Show from Maxwell's equations that the electric and magnetic fields in free space obey a wave equation and deduce the wave speed c in terms of ε_0 and μ_0.

$$\vec\nabla \times \text{(iii)} \longrightarrow \vec\nabla \times (\vec\nabla \times \vec E) = -\vec\nabla \times \frac{\partial \vec B}{\partial t}$$

$$\therefore \vec\nabla(\vec\nabla\cdot\vec E) - \nabla^2 \vec E = -\frac{\partial}{\partial t}(\vec\nabla \times \vec B)$$

0 in vacuum

$$\therefore 0 \quad - \nabla^2 \vec E = -\frac{\partial}{\partial t}\left(\mu_0 \vec J + \mu_0 \varepsilon_0 \frac{\partial \vec E}{\partial t}\right)$$

$$\therefore \nabla^2 \vec E = \mu_0 \varepsilon_0 \frac{\partial^2 \vec E}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 \vec E}{\partial t^2}$$

wave equation, speed $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$

7. [5] Show that ∇ can be replaced by $i\mathbf{k}$ when dealing with a monochromatic plane wave (MCPW) of wavevector $\mathbf{k}$ in complex notation.

$$\vec\nabla \left(e^{i(\vec k\cdot\vec r - \omega t)}\right) = \left(\hat x \frac{\partial}{\partial x} + \hat y \frac{\partial}{\partial y} + \hat z \frac{\partial}{\partial z}\right) e^{i[(k_x x + k_y y + k_z z) - \omega t]}$$

(in Cartesians)

$$= (\hat x\, i k_x + \hat y\, i k_y + \hat z\, i k)\, e^{i(\vec k\cdot\vec r - \omega t)}$$

$$= i\vec k\, e^{i(\vec k\cdot\vec r - \omega t)}$$

8. [10] Consider an MCPW whose electric field is given by $\mathbf{E} = \mathrm{Re}\{E_0 \hat{\mathbf{x}} \exp i(kz - \omega t)\}$. Using one of Maxwell's equations, find the magnetic field $\mathbf{B}$ of the wave.

$$\vec\nabla \times \vec E = -\frac{\partial \vec B}{\partial t} \rightarrow i\vec k \times \vec E = -(-i\omega)\vec B \quad \text{for an MCPW}$$

$$\therefore \vec B = \frac{\vec k \times \vec E}{\omega} = \frac{k\hat z \times E_0 \hat x\, e^{i(kz-\omega t)}}{\omega}$$

$$= \frac{k}{\omega} E_0 \hat y\, e^{i(kz-\omega t)} = \frac{E_0}{c}\hat y\, e^{i(kz-\omega t)}$$

9. [15] Show that the $\mathbf{E}$ and $\mathbf{B}$ fields of this wave satisfy all the remaining Maxwell equations.

$$\vec\nabla\cdot\vec B = i\vec k\cdot\left[\frac{E_0}{c}\hat y\, e^{i(kz-\omega t)}\right] = 0$$

because $\vec k\cdot\hat y = 0$ as $\vec k = k\hat z$

$$\vec\nabla\cdot\vec E = i\vec k\cdot\left[E_0 \hat x\, e^{i(kz-\omega t)}\right] = 0$$

because $\vec k\cdot\hat x = 0$

$$\vec\nabla \times \vec B = i\vec k \times \left[\frac{E_0}{c}\hat y\, e^{i(kz-\omega t)}\right] = \frac{iE_0}{c} k\, \hat z \times \hat y\, e^{i(kz-\omega t)}$$

$$= \frac{ik}{c} E_0 (-\hat x) e^{i(kz-\omega t)}$$

$$= -\frac{i\omega}{c^2} E_0 \hat x\, e^{i(kz-\omega t)}$$

$$= \frac{1}{c^2}\frac{\partial}{\partial t}\vec E$$

as reqd by (iv).

10. [10] A single mode in a laser cavity is a standing wave with electric field $\mathbf{E} = E_0\hat{\mathbf{x}}e^{-i\omega t}\sin kz$. Find the pressure exerted on the laser mirrors.

$$\vec{E} = \frac{E_0\hat{x}}{2i}\left(e^{ikz} + e^{-ikz}\right)e^{-i\omega t}$$ = perfectly reflected MCPW of amplitude $E_0/2$

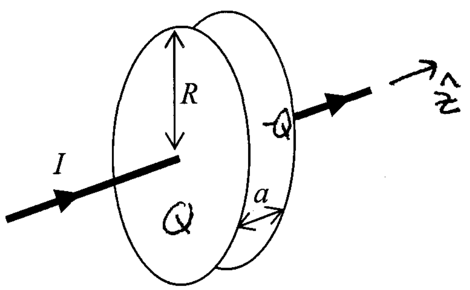
pressure = 2 × incident (reflected) momentum flux density

= 2 × $\langle u_{em}\rangle$, for incident wave

$= 2 \times \frac{\varepsilon_0}{2}\left(\frac{E_0}{2}\right)^2 = \frac{1}{4}\varepsilon_0 E_0^2$

III. A circular parallel-plate capacitor is connected in series with two thin straight wires as shown. The radius of the plates is R, and their separation is $a \ll R$, so that fringing effects and the electric field outside the capacitor may be neglected. A current I is passed along the wires, which lie along the z-axis.

11. [25] Find the electric and magnetic fields in the gap, as a function of the distance r ($< R$) from the axis and at time t. (Assume the charge Q is spread uniformly over the plates and is zero at $t = 0$.)



Charge density on plates is

$$\sigma = \frac{Q}{\pi R^2}$$ where $I = \frac{dQ}{dt}$

$$\therefore \vec{E} = \frac{\sigma}{\varepsilon_0}\hat{z} = \frac{Q}{\pi R^2\varepsilon_0}\hat{z}$$ $Q = It$

By symmetry as usual, $\vec{B} = B(r)\hat{\phi}$

Taking a circular Stokes loop at $r < R$ inside gap,

$$2\pi r B(r) = \int_{\text{disc of radius } r} \vec{\nabla}\times\vec{B}\cdot d\vec{S}$$

Now $\vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$

$$= \mu_0\varepsilon_0\frac{\hat{z}}{\pi R^2\varepsilon_0}\frac{dQ}{dt} = \frac{\mu_0\varepsilon_0 I}{\pi R^2}\hat{z}$$

$$= \pi r^2 \cdot \frac{\mu_0\varepsilon_0 I}{\pi R^2}\hat{z} = \frac{\mu_0 I r}{2\pi R^2}$$

$$\therefore \vec{B} = \frac{\mu_0 I r}{2\pi R^2}\hat{\phi}$$

12. [10] Find the energy density u_{em} and the Poynting vector **S** in the gap.

$$u_{em} = \frac{1}{2\mu_0}B^2 + \frac{\varepsilon_0}{2}E^2 = \frac{1}{2\mu_0}\left(\frac{\mu_0 I r}{2\pi R^2}\right)^2 + \frac{\varepsilon_0}{2}\left(\frac{Q}{\pi R^2 \varepsilon_0}\right)^2 = \frac{\mu_0 I^2 r^2}{8\pi^2 R^4} + \frac{Q^2}{2\pi^2 R^4 \varepsilon_0}$$

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}\left(\frac{Q}{\pi R^2 \varepsilon_0}\hat{z}\right)\times\left(\frac{\mu_0 I r}{2\pi R^2}\hat{\phi}\right) = -\frac{QIr}{(\pi R^2)^2 . 2\varepsilon_0}\hat{r}$$

13. [10] Show from these that energy is conserved locally in the fields.

$$\vec{\nabla}.\vec{S} = \frac{1}{r}\frac{\partial}{\partial r}(rS_r) + 0 = \frac{1}{r}\frac{d}{dr}\left[\frac{-QIr^2}{(\pi R^2)^2 . 2\varepsilon_0}\right] = \frac{-QI}{(\pi R^2)^2\varepsilon_0}$$

$$\frac{\partial u_{em}}{\partial t} = \frac{d}{dt}\left(\frac{Q^2}{2\pi^2 R^4 \varepsilon_0}\right) + 0 = \frac{2Q\dot{Q}}{2(\pi R^2)^2\varepsilon_0} = \frac{QI}{(\pi R^2)^2\varepsilon_0} \quad \therefore \vec{\nabla}.\vec{S} = -\frac{\partial u_{em}}{\partial t}$$

14. [10] Determine the total energy of the fields in the gap, as a function of time.

Energy stored $U = \int u_{em}\, d^3r = \int_0^R 2\pi r a\left(\frac{\mu_0 I^2 r^2}{8\pi^2 R^4} + \frac{Q^2}{2\pi^2 R^4 \varepsilon_0}\right)dr$

$$= 2\pi a\left(\frac{\mu_0 I^2 R^4}{32\pi^2 R^4} + \frac{Q^2 R^2}{4\pi^2 R^4 \varepsilon_0}\right) = \frac{aI^2}{\pi}\left(\frac{\mu_0}{16} + \frac{t^2}{2\varepsilon_0 R^2}\right) \qquad Q = It$$

15. [15] Show that the total power flowing into the gap through an appropriately chosen surface is equal to the rate of increase of energy stored in the gap.

Find energy flowing thr' cylindrical surface (just inside) at edge of capacitor:

inward flux of energy $= -(2\pi Ra)S_r = 2\pi Ra \left.\frac{QIr}{(\pi R^2)^2 . 2\varepsilon_0}\right|_{r=R} = 2\pi Ra . \frac{QI}{\pi^2 R^3 . 2\varepsilon_0}$

$$= \frac{aQI}{\pi R^2 . \varepsilon_0}$$

$$\frac{dU}{dt} = 0 + \frac{2aI^2 t}{\pi . 2\varepsilon_0 R^2} = \frac{aQI}{\pi R^2 \varepsilon_0} = \text{inward flux of energy}$$

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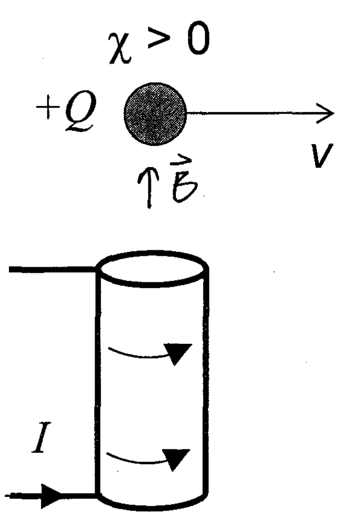
16. [5] Find the total momentum contained in the fields in the gap.

Momentum density $= \vec{S}/c. = \vec{p}_{em}$

Since $\vec{S}$ is radially symmetric, $\int \vec{p}_{em}\, d^3r = 0.$

II. A paramagnetic ($\chi > 0$) metallic sphere with a net positive charge $+Q$ is moving past the end of a solenoid carrying current I, as sketched below.

17. [25] What electromagnetic forces act on the sphere, and in what directions do they each act? For each, indicate the relevant principle(s) and equations.



(1) Lorentz force $\vec{F}_L = +Q(\vec{v} \times \vec{B})$

$\vec{B}$ is upwards by r.h. rule

$\therefore \vec{F}_L$ is out of page.

(2) Drag force due to eddy currents induced by changing B-field seen by sphere. By Lenz's law, force opposes change

$\therefore \vec{F}_{drag}$ is to the left, opposite to $\vec{v}$.

(3) Magnetic force:

magnetic energy $U = -\vec{M}.\vec{B}$

$\vec{M} = \chi \vec{H} = \chi \mu_0 B \qquad \therefore U = -\mu_0 \chi B^2$

$\therefore U$ decreases as B increases if $\chi > 0$

$\therefore$ sphere is pulled towards solenoid

$\vec{F}_{mag}$ is downwards.