323 First Midterm Solutions

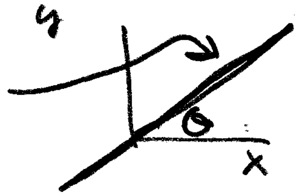
1. (a) (10) The electric field associated with an electromagnetic plane wave moving in the $+z$-direction, in a medium with index of refraction $n = 1$, is given by

$$\mathbf{E}^{(\lambda)}(z,t) = E_0\; e^{i(kz-\omega t)}\;\; (\hat{\mathbf{e}}_x + \lambda \hat{\mathbf{e}}_y)\,,$$

where λ is a given real number. Determine the polarization of the wave.

$$\frac{E_x^{\lambda}(z,t)}{E_y^{\lambda}(z,t)} = \lambda = \text{real, and time independent}$$

wave is linearly polarized in a direction given by



with $\tan\theta = \lambda$

(b) (15) Determine the intensity of the wave of part (a).

$$\vec{B} = \frac{\vec{k}}{\omega} \times \vec{E} = \frac{E_0}{c}\, e^{i(kz-\omega t)}\; \hat{z} \times (\hat{e}_x + \lambda \hat{e}_y)$$

$$= \frac{E_0}{c}\, e^{i(kz-\omega t)}\; (+\hat{y} - \lambda \hat{x})$$

$$I = \hat{k} \cdot \langle \vec{S} \rangle = \frac{\hat{k}}{2} \cdot \left(\vec{E} \times \frac{\vec{B}^*}{\mu_0} \right) = \frac{1}{2\mu_0 c} |E_0|^2 (1+\lambda^2)$$

$$= \frac{\varepsilon_0 c}{2} |E_0|^2 (1+\lambda^2)$$

2. Consider light of angular frequency ω traveling from $z = -\infty$ incident normally on a plate of glass with thickness a and real index of refraction n. The plate is parallel to the xy plane, with one face at $z = 0$ and the other at $z = a$. The index of refraction is unity for $z < 0, z > a$. The electromagnetic field for $z < 0$ is a superposition of incident and reflected waves. In the region $0 \le z \le a$ there are waves moving in the positive and negative z directions. In the region $z > a$ there is only a transmitted wave.

(a) (15) Write equations for $\mathbf{E}(z,t)$ and $\mathbf{B}(z,t)$ in all three regions of z. The direction of polarization of the incident wave is $\hat{\mathbf{j}}$, and its amplitude is E_0, a real number. Express your answers in terms of 4 unknown electric field amplitudes.

$z<0$ $\quad \vec{E} = E_0 \hat{j}\, e^{i(kz-\omega t)} + E_R \hat{j}\, e^{i(kz-\omega t)}$ $\qquad k = \frac{\omega}{c}$

$\vec{B} = -\frac{E_0}{c} \hat{\imath}\, e^{i(kz-\omega t)} + \frac{E_R}{c} \hat{\imath}\, e^{i(-kz+\omega t)}$

$0<z<a$ $\quad \vec{E} = E_1 \hat{j}\, e^{i(k_1 z-\omega t)} + E_2 \hat{j}\, e^{-i(k_2 z+\omega t)}$ $\qquad k_2 = \frac{\omega}{c} n = nk$

$\vec{B} = -\frac{E_1 n}{c} \hat{\imath}\, e^{i(k_1 z-\omega t)} + \frac{E_2 n}{c} \hat{\imath}\, e^{-i(k_2 z+\omega t)}$

$0<z\,a$ $\quad \vec{E} = E_t \hat{j}\, e^{i(kz-\omega t)}$ $\quad \vec{B} = -\frac{E_t}{c} \hat{\imath}\, e^{i(kz-\omega t)}$

(b) (10) List the quantities that are continuous at $z = 0$ and $z = a$

$(\hat{z})$ Normal components of $\vec{D}, \vec{B}$
(x,y) tangential components of $\vec{E}, \vec{H}$
$\Big\}$ continuous at both $z=0, a$

(c) (15) Write the boundary conditions that the amplitudes of the electric field obey at $z = 0, a$.

$z=0$ $\quad$ tang E continuous $\qquad E_0 + E_R = E_1 + E_2$

tang H cont" $\qquad -\frac{E_0}{c} + \frac{E_R}{c} = \left(-\frac{E_1}{c} + \frac{E_2}{c}\right) n$

$z=a$ $\quad$ E tang: $\qquad E_1 e^{ikan} + E_2 e^{-ikan} = E_t e^{ika}$

tang H $\qquad n\left(-\frac{E_1}{c} e^{ikan} + \frac{E_2}{c} e^{-ikan}\right) = -\frac{E_t}{c} e^{ika}$

3. A plasma is an ionized gas consisting at least partly of free electrons and positive ions; it is therefore a conducting material. The sun and stars are largely plasmas. Consider the classical model discussed in class and in sect. 9.4.3, with $\gamma = 1/\tau$ and the electrons are not bound. Take the electric field to have the form $\mathbf{E}(t) = \mathbf{E}_0 e^{-i\omega t}$.

(a) (15) The density of electrons in the plasma is N, and the current density $\mathbf{J} = -eN\mathbf{v} = \sigma\mathbf{E}$. Show that the conductivity σ is given by $\sigma(\omega) = \frac{i(\omega e^2/m)\,N}{\omega^2+i\omega\gamma} = \frac{i\epsilon_0\Omega^2}{\omega+i\gamma}$, with $\Omega^2 \equiv Ne^2/(m\epsilon_0)$.

$$m\ddot{\vec{r}} + m\gamma\dot{\vec{r}} = -e\vec{E} \rightarrow (-m\omega^2 - i m\gamma\omega)\vec{r} = -e\vec{E}$$

$$\vec{v} = -i\omega\vec{r} = +iew\vec{E}/(-m)(\omega^2+i\gamma\omega) = -i(e\vec{E}/m)/(\omega+i\gamma)$$

$$\sigma = -eN(-ie/m)/(\omega+i\gamma)$$

$$= \frac{i e^2 N/m}{\omega+i\gamma} = \frac{i\Omega^2\epsilon_0}{\omega+i\gamma}$$

(b)(10) For a dilute plasma the damping factor γ is small, so for the remainder of this problem, take $\gamma = .01\Omega$, and for a typical plasma $\Omega = 1.5 \times 10^{10}$ rad/s. For what frequencies is $\mathrm{Re}(\sigma) \gg \mathrm{Im}(\sigma)$? For such frequencies, is the wave damped or oscillatory? Explain.

$$\sigma = \frac{\Omega^2\epsilon_0}{\omega^2+\gamma^2}\, i/(\omega - i\gamma) = \frac{\Omega^2\epsilon_0}{\omega^2+\gamma^2}(\gamma+i\omega)$$

$$\frac{\mathrm{Re}\,\sigma}{\mathrm{Im}\,\sigma} = \frac{\gamma}{\omega} \gg 1 \text{ if } \gamma \gg \omega \text{ or } \omega \ll \gamma = 1.5\times10^8 \text{ rad/s}$$

For $\mathrm{Re}\,\sigma \gg \mathrm{Im}\,\sigma$, the plasma behaves as an ordinary conductor and the wave is damped.

(c)(10) For what frequencies is $\mathrm{Re}(\sigma) \ll \mathrm{Im}(\sigma)$? For such frequencies, is the wave damped or oscillatory? Explain.

$\mathrm{Re}\,\sigma \ll \mathrm{Im}\,\sigma$ if $\omega \gg \gamma = 1.5\times10^8$ rad/s

if $\omega \gg \gamma$, $\omega > \Omega$ and the waves propagate without damping