Name __________solutions________

Electrodynamics, Physics 323 **First midterm** **8.20 am, April 23, 2004**
Spring 2004 Instructor: David Cobden

You have 60 minutes. End on the buzzer at 9.20. Answer all 11 questions.

Write your name on every page and your ID on the first page.

Write all your working on these question sheets. Use this cover page for extra working (you might get credit for it.)

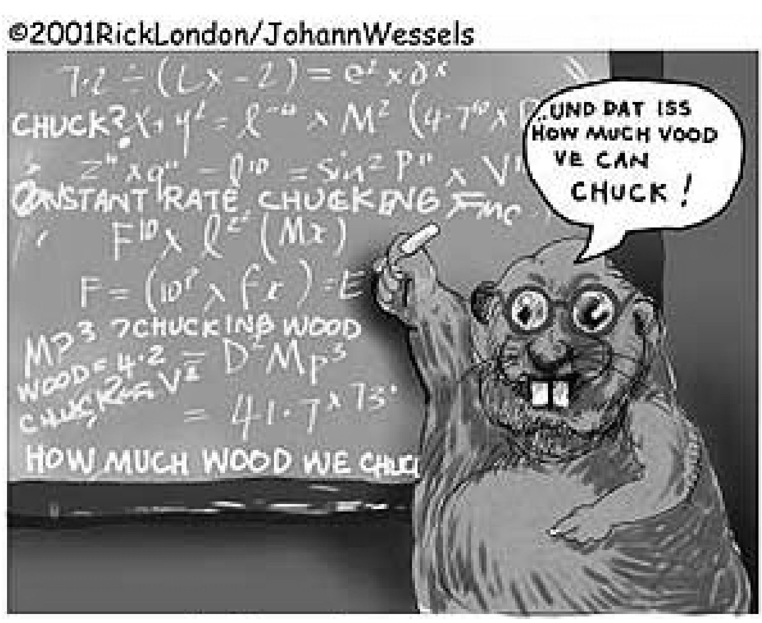
It is important to show your calculation or derivation. You won't get full marks just for stating the correct answer if you don't show how you get it.

Watch the blackboard for corrections or clarifications during the exam.

This is a **closed book** exam. **No notes allowed. No calculators.**

Do not turn this page until I say 'go'.

In a dilute plasma or in a waveguide, $w^2 = w_p{}^2 + c^2k^2$



 Name solutions

I. A plane electromagnetic wave, of electric field amplitude E_{0I}, is normally incident on a dielectric boundary across which the refractive index changes from n_1 to n_2, where $n_1 > n_2$.

1. [5] What is the magnitude of the magnetic field in the incident wave, B_{0I}?

$$\vec{B} = \frac{\vec{k} \times \vec{E}}{\omega} \rightarrow B_{0I} = \frac{k}{\omega} E_{0I} = \frac{E_{0I}}{v} = \frac{n_1 E_{0I}}{c}$$

2. [5] State the relationship between the incident, reflected and transmitted amplitudes, E_{0I}, E_{0R} and E_{0T}, resulting from the boundary condition on electric field.

$$E_{0I} + E_{0R} = E_{0T} \qquad (E_{\parallel} \text{ is continuous})$$

3. [15] Assuming $\mu_1 = \mu_2$, the amplitude of the reflected wave is $E_{0R} = \frac{(n_1 - n_2)}{(n_1 + n_2)} E_{0I}$.

Using conservation of energy, or otherwise, derive the amplitude of the transmitted wave, E_{0T}.

Cons. of energy $\Rightarrow$ $T = 1 - R$

ie $v_2 \varepsilon_2 E_{0T}^2 = v_1 \varepsilon_1 (E_{0I}^2 - E_{0R}^2)$

$$\therefore E_{0T}^2 = \frac{v_1 \varepsilon_1}{v_2 \varepsilon_2} \left[1 - \left(\frac{E_{0R}}{E_{0I}} \right)^2 \right] E_{0I}^2$$

$$= \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} \left[1 - \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2 \right] E_{0I}^2$$

$$= \frac{n_1}{n_2} \frac{4 n_1 n_2}{(n_1 + n_2)^2} E_{0I}^2 = \frac{4 n_1^2}{(n_1 + n_2)^2} E_{0I}^2$$

$$\therefore E_{0T} = \left(\frac{2 n_1}{n_1 + n_2} \right) E_{0I}$$

OR :

$$E_{0T} = E_{0I} + E_{0R}$$

$$= E_{0I} \left[1 + \left(\frac{n_1 - n_2}{n_1 + n_2} \right) \right] = \left(\frac{2 n_1}{n_1 + n_2} \right) E_{0I}$$

 Name solutions

II. Dilute plasma.

4. [10] Starting from the appropriate Maxwell equation(s) in a linear conducting medium, derive the dispersion relation: $k^2 = \mu\varepsilon\omega^2 + i\sigma\mu\omega$.

$$\vec{\nabla}\times\vec{B} = \mu\frac{\partial\vec{D}}{\partial t} + \mu_0\vec{J} \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\cdot\vec{E} = \rho/\varepsilon$$

$$\therefore \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) = -\mu\frac{\partial^2\vec{D}}{\partial t^2} - \mu\frac{\partial\vec{J}}{\partial t}$$

Linear conducting medium: $\vec{D} = \varepsilon\vec{E}$, $\vec{J} = \sigma\vec{E}$, $\rho = 0$

$$\vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{E}}_{0}) - \nabla^2\vec{E} = -\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} - \mu\sigma\frac{\partial\vec{E}}{\partial t} \qquad \therefore \nabla^2\vec{E} = \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} + \mu\sigma\frac{\partial\vec{E}}{\partial t}$$

Put $\vec{E} = \vec{E}_0 e^{i(kz-\omega t)} \rightarrow k^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega$

5. [15] Show that electromagnetic waves cannot propagate in a dilute plasma for $\omega < \omega_p$, where the plasma frequency is $\omega_p = (Nq^2/m\varepsilon_0)^{1/2}$. You should derive this result by considering a free charged particle oscillating in response to the electric field. Define all the terms in the equation for ω_p.

Eqn of motion: $m\ddot{x} = qE_0e^{-i\omega t}$ (no damping, $\gamma\omega \ll 1$)

$$\therefore \dot{x} = \frac{-qE}{i\omega m}$$

q = charge per particle
m = mass " "

Current density $J = Nq\dot{x}$ $\quad N$ = particle density

$$= -\frac{Nq^2E}{im\omega} = \sigma E$$

$$\therefore \sigma = \frac{iNq^2}{m\omega}$$

$$\therefore k^2 = \mu_0\varepsilon_0\omega^2 + i\mu_0\left(\frac{iNq^2}{m\omega}\right)\omega$$

Assumes $\mu = \mu_0$, $\varepsilon = \varepsilon_0$ (no bound charges)

$$\therefore k^2 = \mu_0\varepsilon_0\omega^2 - \frac{\mu_0Nq^2}{m} = \mu_0\varepsilon_0(\omega^2 - \omega_p^2) \;;\; \omega_p^2 = \frac{Nq^2}{m\varepsilon_0}$$

for $\omega < \omega_p$, k is purely imaginary
$\rightarrow \vec{E}$ decays exponentially with no oscillations
$\Rightarrow$ no wave propagation.

 Name solutions

6. [10] At what speed, in terms of c, does a radio pulse of central frequency $\omega = 2\omega_p$ propagate through the plasma?

$$\omega^2 = \omega_p^2 + c^2 k^2$$

Pulse goes at group velocity, $v_g = \frac{d\omega}{dk} = c^2 \frac{k}{\omega} = c\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}$

$$= c\sqrt{1-\left(\tfrac{1}{2}\right)^2}$$

$$= \frac{\sqrt{3}}{2} c.$$

III. The inside of a (square) metal waveguide is defined by the region $0 < x < a$, $0 < y < a$. *Consider only its TM modes.*

7. [5] State B_z and show that it matches the boundary conditions on **B**.

$B_z = 0$ b.c. is $B_\perp = 0$ ie $B_x = B_y = 0$ at surface.

B_z is $B_{\parallel}$ which can be anything! $(\vec{B}_{\parallel} = \hat{n} \times \vec{k})$

8. [10] We search for waveforms propagating in the positive z direction, of the form $\mathbf{E}(x,y)e^{i(kz-\omega t)}$. From Maxwell's wave equation for **E**, derive a differential equation for $E_z(x,y)$ which involves ω but not time.

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \qquad \therefore \nabla^2 E_z = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$$

Put $E_z = E_z(x,y)e^{i(kz-\omega t)}$ in here

$$\rightarrow \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} - k^2 E_z = -\frac{\omega^2}{c^2} E_z$$

$$\therefore \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c^2} - k^2\right] E_z = 0$$

 Name solutions

9. [5] State the boundary conditions on $E_z(x,y)$.

$E_z = 0$ at $x=0$, $x=a$, $y=0$ and $y=a$.

($E_{\parallel} = 0$ at surface of conductor)

10. [15] Determine the set of functions $E_z(x,y)$, (corresponding to the TM_{mn} modes), that satisfy the equation from Q.8 and the boundary conditions from Q.9.

Separate variables: put $E_z(x,y) = X(x)Y(y)$

$$\rightarrow \frac{X''}{X} + \frac{Y''}{Y} + \frac{\omega^2}{c^2} - k^2 = 0$$

$\therefore X(x) = \sin\frac{n\pi x}{a}$ $\quad Y(y) = \sin\frac{m\pi y}{a}$ (matching b.c.'s)

$$\therefore -\left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{a}\right)^2 + \frac{\omega^2}{c^2} - k^2 = 0$$

$$\therefore \omega^2 = c^2k^2 + \frac{\pi^2c^2}{a^2}(n^2+m^2)$$

$E_z = (\text{const}) \times \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{a}$

gives TM_{nm} mode

Lowest nonvanishing mode is $n=m=1$

11. [5] Find the cutoff frequency for the lowest TM mode.

$$\omega_{11} = \frac{\pi c}{a}\sqrt{1^2+1^2} = \sqrt{2}\,\frac{\pi c}{a}$$