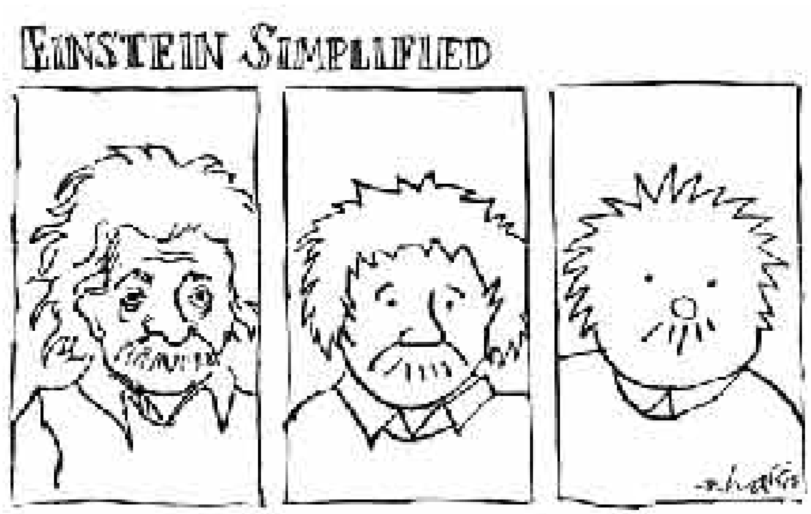
Name ____________solutions______________

Electrodynamics, Physics 323 **First midterm** **8.20 am, April 25, 2005**
Spring 2005 Instructor: David Cobden

You have 60 minutes. End on the buzzer at 9.20. Answer all questions.
You are strongly recommended to read through all the questions before you begin.
Write your name on every page and your ID on the first page.
Watch the blackboard for corrections or clarifications during the exam.
In this exam you are allowed **no books, no note sheets, and no calculators!**

Write all your working on these question sheets. You will get credit for it. It is important to show your calculation or derivation. You usually won't get full marks just for stating the correct answer.

Do not turn this page until the buzzer goes.



 Name solutions

I. Consider the following equation for the time-varying electric field in a material:

$$\nabla^2 \mathbf{E} = \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{E}}{\partial t}, \qquad (1)$$

where μ, ε and σ are scalars or second-rank tensors.

1. [24] Indicate whether *each* of the following statements about Eq. (1) is true or false: (write T or F)

F a. It is valid only in metallic (reasonably conducting) materials.

F b. It is valid only at frequencies between some characteristic lower and upper limits.

T c. It is valid only in linear materials.

F d. It is valid only in isotropic materials.

T e. It is valid only if the net space charge $\rho(\mathbf{r})$ is constant or zero.

F f. It is not valid in a dilute plasma.

T g. At finite frequency ω the distinction between σ and ε is arbitrary.

T h. The magnetic field obeys exactly the same equation.

2. [6] Find the general dispersion relation $k(\omega)$ for a monochromatic plane wave that obeys this equation, indicating all possible origins of the freqency dependence of k.

Put $\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$ into (1):

$$\rightarrow -k^2\vec{E} = -\omega^2\mu\varepsilon\vec{E} - i\omega\mu\sigma\vec{E}$$

$$\therefore k^2 = \mu(\omega)\,\varepsilon(\omega)\,\omega^2 + i\mu(\omega)\,\sigma(\omega)\,\omega$$

3. [10] Using one of Maxwell's equations, find an equation relating the magnetic and electric fields for the plane wave, and deduce the impedance $Z = E/H$ of the material in terms of the parameters in Eq. (1).

If $\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$ (plane wave)

$$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E} \qquad \therefore -i\omega\vec{B} = -i\vec{k}\times\vec{E} \qquad \therefore \vec{B} = \frac{\vec{k}\times\vec{E}}{\omega}$$

$\vec{\nabla}\cdot\vec{E} = 0 \rightarrow \vec{k}\cdot\vec{E} = 0 \rightarrow \vec{E}$ is perp to $\vec{k}$

$$\therefore |B| = \frac{|k||E|}{\omega}$$

$$Z = \frac{|E|}{|H|} = \mu\frac{|E|}{|B|} = \frac{\mu\omega}{|k|} = \frac{\mu\omega}{(\mu\varepsilon\omega^2 + i\mu\sigma\omega)^{1/2}} = \left(\frac{\mu}{\varepsilon + i\sigma/\omega}\right)^{1/2}$$

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4. [10] For a dielectric ($\sigma = 0$), show that the wave phase velocity at high frequencies is $v_{ph} \approx c/\mathrm{Re}[\sqrt{\varepsilon_r}]$.

Now $k^2 = \mu \varepsilon \omega^2$ $\therefore k = \sqrt{\mu\varepsilon}\,.\,\omega = \sqrt{\mu_0\varepsilon_0\varepsilon_r}\,.\,\omega = \sqrt{\varepsilon_r}\,.\,\frac{\omega}{c}$

$$= k' + \frac{i\alpha}{2} \qquad k' = \mathrm{Re}[k]$$

$$V_{ph} = \frac{\omega}{k'} = \frac{\omega}{\mathrm{Re}[k]} = \frac{\omega}{\mathrm{Re}[\sqrt{\varepsilon_r}\,.\,\frac{\omega}{c}]} = \frac{c}{\mathrm{Re}[\sqrt{\varepsilon_r}]}$$

5. [5] For a conductor, Eq. (1) implies a characteristic timescale τ for motion (relaxation) of free charge. What is τ, in terms of the parameters in the equation?

Dimensionally, $\mu\varepsilon\frac{E}{t^2} = \mu\sigma\frac{E}{t}$ $\therefore$ timescale (t) $\tau = \frac{\varepsilon}{\sigma}$

6. [10] Show that the phase velocity of monochromatic plane waves in a conductor with $\varepsilon_r = 1 = \mu_r$ in the limit $\omega\tau \ll 1$ is $(2\omega\tau)^{1/2}c$.

$$k^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega = \mu\sigma\tau\omega^2 + i\mu\sigma\omega = \mu\sigma\omega(\omega\tau + i)$$

For $\omega\tau \ll 1$, $k^2 \doteq i\mu\sigma\omega$ $\therefore k = \frac{1+i}{\sqrt{2}}\sqrt{\mu\sigma\omega} = \frac{1+i}{\delta}$; $\delta = \sqrt{\frac{2}{\mu\sigma\omega}}$

again $V_{ph} = \frac{\omega}{\mathrm{Re}[k]} = \frac{\omega}{1/\delta} = \sqrt{\frac{2\omega}{\mu\sigma}} = \sqrt{\frac{2\omega\tau}{\mu\varepsilon}} = \sqrt{2\omega\tau}\,\frac{c}{\sqrt{\mu_r\varepsilon_r}}$

$$= \sqrt{2\omega\tau}\,.\,c \quad \text{if } \mu_r = 1 = \varepsilon_r$$

7. [10] Show that information is carried at much less that the speed of light by such waves.

Information moves at group velocity $V_g = \frac{d\omega}{dk'}$ $\approx \mathrm{Re}[k]$

Here $\frac{\omega}{k'} = V_{ph} = \sqrt{2\omega\tau}\,.\,c$

$$\therefore \omega^{1/2} = \sqrt{2\tau}\,ck'$$

$$\therefore \frac{1}{2\omega^{1/2}}\frac{d\omega}{dk'} = (2\tau)^{1/2}c \qquad \therefore V_g = \frac{d\omega}{dk'} = (2\tau)^{1/2}\,2\omega^{1/2}c$$

$$= 2\sqrt{2\omega\tau}\,c$$

$$\ll c$$

as $\omega\tau \ll 1$

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8. [10] The conductivity of silver is about $8\times10^{7}\ \Omega^{-1}\mathrm{m}^{-1}$. *Estimate* the minimum thickness of the silver film needed on the back of a mirror. (Take $\mu_0 \approx 10^{-6}\ \mathrm{Hm}^{-1}$).

At optical frequencies $f \sim 10^{15}$ Hz

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}} \quad \text{from above}$$

$$\simeq \sqrt{\frac{2}{(10^{-6})(8\times10^{7})(2\pi\times10^{15})}}$$

$$\simeq \frac{1}{\sqrt{24\times10^{16}}} = \frac{10^{-8}\,m}{\sqrt{24}} \simeq \frac{10^{-8}\,m}{5} = 20\,Å$$

$\therefore$ need thickness say 5δ $= 100\,Å$.

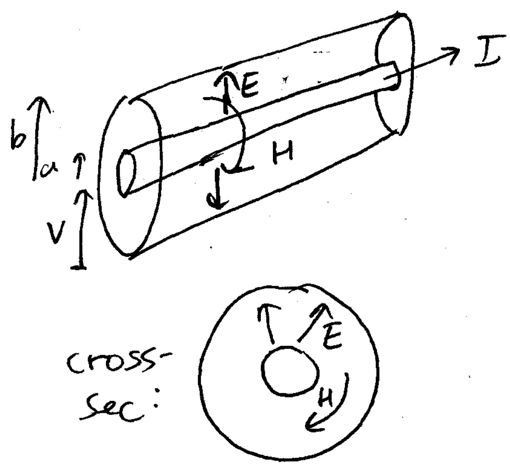
II. A transmission line consists of two coaxial conductors. The space between them, which occupies the region $a < r < b$ where r is the distance from the axis, is filled with a linear dielectric of dielectric constant ε_r and negligible magnetic susceptibility.

9. [5] State the dispersion relation for a TEM mode travelling along the line.

$$v_{ph} = \frac{\omega}{k} = \frac{c}{\sqrt{\varepsilon_r}}$$ (no cutoff; same as for MCPWs)

10. [20] Find the impedance $Z = V/I$ for the TEM mode.

TEM → field patterns in cross-sec are same as for static case (dc coax cable) (× traveling wave $e^{i(kz-\omega t)}$ along cable)



λ = charge/unit length

$$E = \frac{\lambda}{2\pi\varepsilon r} \quad \text{by Gauss in dielectric}$$

$$V = \int_b^a (-E)\,dr = \frac{\lambda}{2\pi\varepsilon}\ln\frac{b}{a}$$

$I = \lambda v$ (if you like, $I = 2\pi a K = 2\pi a H_{\parallel}(a) = 2\pi a \frac{E(a)}{Z} = 2\pi a \cdot \frac{\lambda}{2\pi\varepsilon a}\cdot\sqrt{\frac{\varepsilon}{\mu}} = \lambda v$)

$$\therefore \frac{V}{I} = \frac{\frac{\lambda}{2\pi\varepsilon}\ln\frac{b}{a}}{\lambda v}$$

$$= \frac{\ln\frac{b}{a}}{2\pi\varepsilon v} = \frac{\ln\frac{b}{a}}{2\pi}\sqrt{\frac{\mu}{\varepsilon}} = \frac{\ln\frac{b}{a}}{2\pi\sqrt{\varepsilon_r}} Z_0$$