Name solutions

Electrodynamics, Physics 323 **First midterm** **8.20 am, Friday May 4, 2007**
Spring 2007 Instructor: David Cobden

Do not turn this page until the buzzer goes at 8.20. You must hand your exam to me before I leave the room at 9.25.

Attempt all the questions.

Please write your name on every page and your SID on the first page.

Write all your working on these question sheets. Use this front page for extra working. It is important to show your calculation or derivation. Some of the marks are given for showing clear and accurate working and reasoning.

Watch the blackboard for corrections or clarifications during the exam.

This is a closed book exam. ***No books, notes or calculators allowed.***

 Name solutions

1. [10] State Maxwell's equations in vacuum.

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

2. [10] For a monochromatic plane wave, show that **E**, **B** and **k** are mutually perpendicular.

$$\vec{\nabla}\cdot\vec{E} = 0 \rightarrow i\vec{k}\cdot\vec{E} = 0 \quad \therefore \vec{E}\perp\vec{k} \qquad \vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \rightarrow i\vec{k}\cdot\vec{B} = 0 \quad \therefore \vec{B}\perp\vec{k}$$

$$\vec{\nabla}\times\vec{E} = i\vec{k}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = i\omega\vec{B} \quad \therefore \vec{B} = \frac{\vec{k}\times\vec{E}}{\omega} \quad \therefore \vec{B} \perp r\ \vec{E}$$

3. [3] A certain laser beam can be treated as a monochromatic plane wave with $\mathbf{k} = k\hat{\mathbf{z}}$ and electric field amplitude $\mathbf{E}_0 = E(\hat{\mathbf{x}} + 2i\hat{\mathbf{y}})$. What kind of polarization is this?

Elliptical.

4. [12] The laser is reflected normally from a perfectly conducting surface. Find the average momentum density in the incident wave and hence the pressure exerted on the surface.

$$\langle S\rangle = \left\langle \frac{1}{\mu_0}\left|\vec{E}\times\vec{B}\right|\right\rangle = \frac{E_0B_0}{2\mu_0} = \frac{E_0^2}{2\mu_0 c} = \frac{\varepsilon_0E_0^2c}{2}$$

= energy / unit time / unit area

$$E_0^2 = E^2(1^2+2^2) = 5E^2$$

Perfect reflection $\Rightarrow$ pressure = 2 × incident momentum flux / unit time

$$\left(\frac{\text{Momentum flux}}{\text{unit time}}\right) = \left(\frac{\text{energy flux}}{\text{unit time}}\right)/c = \frac{\langle S\rangle}{c}$$

$$\therefore \text{pressure} = 2\cdot\frac{\varepsilon_0E_0^2}{2} = \varepsilon_0E_0^2 = 5\varepsilon_0E^2$$

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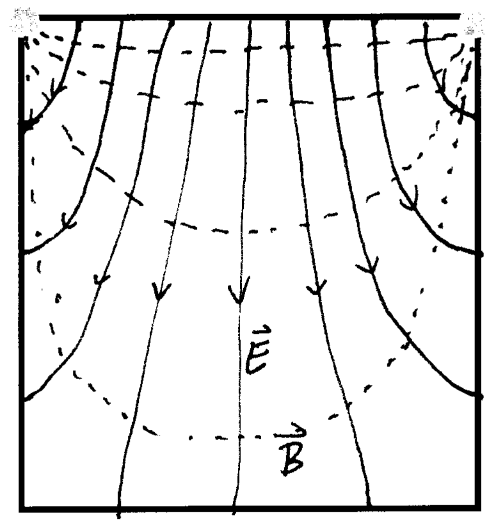
5. [6] A long, empty, metal tube has a square cross-section, of side a. On close inspection it turns out that one of the four sides of the tube is actually a strip of metal which is glued using nonconductive epoxy to the rest, which looks a bit like a gutter. The cross-section is indicated in the diagram. What is the lowest frequency allowed for a propagating electromagnetic mode inside the tube if there is no electrical contact between the strip and the gutter, and why?



2 conductors → TEM modes are allowed, with $\omega = ck$

∴ lowest freq allowed is dc.

6. [10] At a frequency low enough that only this one mode can propagate, *sketch on the diagram* the electric and magnetic field lines inside at some instant in time. Remember to take into account the boundary conditions on **E** and **B**. Also, what can you say about the field components along the tube?

TEM: $E_z = B_z = 0$ at surface, $E_{\parallel} = 0$ $B_{\perp} = 0$ $\vec{B} = \frac{\vec{k} \times \vec{E}}{\omega}$

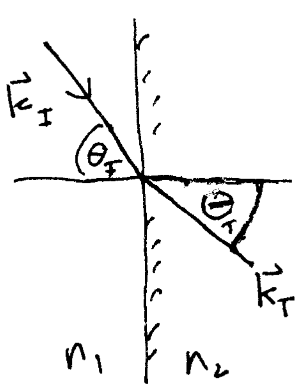
7. [5] Suppose that somewhere along the tube the strip actually makes electrical contact with the gutter. What effect will this have on propagation at this low frequency, and why?

At that place there is only one conductor → no TEM mode. A wave at low frequency will therefore be reflected.

8. [5] Is it still possible for the tube to carry waves at higher frequencies, and why?

Yes - there are still TE and TM modes, so above their cutoff frequencies the wave can propagate.

9. [12] By considering the phase variation of incident and transmitted waves over a planar interface between media with different refractive indices, derive Snell's law of refraction.



We must have $\vec{k}_I \cdot \vec{r} = \vec{k}_T \cdot \vec{r}$ over the boundary

$$k_I \sin\theta_I = k_T \sin\theta_T$$

$V_1 = \frac{c}{n_1}$ $\quad$ ∴ $\frac{\omega}{V_1} \sin\theta_1 = \frac{\omega}{V_2} \sin\theta_2$

$V_2 = \frac{c}{n_2}$ $\quad$ ∴ $n_1 \sin\theta_1 = n_2 \sin\theta_2$

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10. [10] By dimensional analysis, or otherwise, find the form of the relationship between the square of the plasma frequency ω_p^2 and the electron concentration N, the charge $e = 1.6 \times 10^{-19}$ C, the electron mass $m = 10^{-30}$ kg, and $\varepsilon_0 \approx 10^{-11}$ Fm^{-1}.

$$\omega_p^2 = N^a m^b \left(\frac{e^2}{\varepsilon_0}\right)^c \leftrightarrow \frac{1}{L^{3a}} M^b (\text{energy} \times L)^c = \frac{M^b}{L^{3a}}\left(\frac{ML^3}{T^2}\right)^c = \frac{1}{T^2}$$

coe powers of Time $\rightarrow c = 1$

powers of $L \rightarrow a = c = 1$

powers of $M \rightarrow b + c = 0 \therefore b = -1$ $\quad \therefore \omega_p^2 = \frac{Ne^2}{m\varepsilon_0} \times$ number

11. [10] The ionosphere carries a free electron density $N \approx 10^{11}$ m^{-3}. *Estimate* the shortest wavelength of radio waves that can be channeled around the earth by reflection from the ionosphere.

$\omega^2 = c^2k^2 + \omega_p^2$ $\rightarrow$ no propagation for $\omega < \omega_p$

$$\lambda > \frac{2\pi c}{\omega_p}$$

$\therefore$ reflection occurs if $\lambda > 2\pi c \sqrt{\frac{m\varepsilon_0}{Ne^2}}$

$$\simeq \frac{6 \times 3\times10^8\,\text{ms}^{-1}}{1.6\times10^{-19}\,\text{C}} \times \left(\frac{10^{-30}\,\text{kg} \times 10^{-11}\,\text{Fm}^{-1}}{10^{11}\,\text{m}^{-3}}\right)^{1/2} \simeq 10^{1+8+19} \times (10^{-30-11-11})^{1/2} = 10^{28} \times 10^{-26} = 100\,\text{m}.$$

12. [12] By considering the **E** and **B** fields, determine what distribution of charge and/or current generates the following potentials *in cylindrical coordinates*:

$$\mathbf{A} = \frac{I\mu_0 \hat{\mathbf{z}} \ln(r/b)}{2\pi} - \frac{Q\hat{\mathbf{r}}t}{r}, \quad V = Q\ln(r/a).$$

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t} = \underbrace{-Q\vec{\nabla}\ln\left(\frac{r}{a}\right)}_{\text{potential of line charge}} + \underbrace{Q\frac{\hat{r}}{r}}_{\text{field of same line charge}} = 0$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$$

(what else could it be?)

ie we have current I along the $r=0$ axis and $\rho = 0$.